Flipping Physics Lecture Notes:

Deriving the Work-Energy Theorem using Calculus

Yes, this video uses calculus, which is not a part of an algebra based course, however, sometimes it is useful to do math which is above your pay grade, just to see what it looks like.

The definition of work using an integral is: $W = \int_{x_i}^{x_f} F\,dx$

Which is read, Work equals the integral from position initial position to final position of force with respect to position. FYI: because x means position, dx means "with respect to position". For our derivation we will be working with the *net* work and the *net* force. So the equation is: $W_{net} = \int_{x_i}^{x_f} F_{net}\,dx$ and we know, according to Newton's Second Law, that $\sum \vec{F} = m\vec{a}$ therefore $W_{net} = \int_{x_i}^{x_f} (ma)\,dx$

(Let's drop the vector symbol over the acceleration because everything here will be in the x direction.)

Mass is a scalar so it can be removed from the integral: $W_{net} = m\int_{x_i}^{x_f} (a)\,dx$

We can't take this integral yet because acceleration is not with respect to position.

The equation for average acceleration is $a_{average} = \frac{\Delta v}{\Delta t}$, however, we are interested in the calculus version of instantaneous acceleration which is the derivative of velocity with respect to time or $a_{instantaneous} = \frac{dv}{dt}$, which we can substitute in. $\Rightarrow W_{net} = m\int_{x_i}^{x_f} \left(\frac{dv}{dt}\right) dx$ However, we still can't integrate this with respect to position.

It turns out that $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$ therefore $\Rightarrow W_{net} = m\int_{x_i}^{x_f} \left(\frac{dv}{dx}\frac{dx}{dt}\right) dx$

And we can do some canceling out and rearranging: $\Rightarrow W_{net} = m\int_{x_i}^{x_f} \left(\frac{dv}{\cancel{dx}}\frac{dx}{dt}\right) \cancel{dx} = m\int_{v_i}^{v_f} \left(\frac{dx}{dt}\right) dv$

When we changed what we were taking the integral with also needed to change the initial and final limits. That is why we are now taking the integral from the initial velocity to the final velocity.

The equation for average velocity is $v_{average} = \frac{\Delta x}{\Delta t}$, however, we need the calculus version of instantaneous velocity which is the derivative of position with respect to time is velocity:

$v_{instantaneous} = \frac{dx}{dt}$

Therefore: $\Rightarrow W_{net} = m\int_{v_i}^{v_f}\left(v\right)dv$ and the integral of this equation is: $\Rightarrow W_{net} = m\left[\frac{v^2}{2}\right]_{v_i}^{v_f}$

Read: The net work equals the mass of the object times the velocity of the object squared divided by two from velocity initial to velocity final. Which works out to be: $\Rightarrow W_{net} = \frac{1}{2}mv_f^{\ 2} - \frac{1}{2}mv_i^{\ 2}$

This is where the definition of Kinetic Energy comes from: Kinetic Energy is defined as $\frac{1}{2}mv$

Therefore the net work equation is: $\Rightarrow W_{net} = KE_f - KE_i = \Delta KE$

Notice how, unlike $ME_i = ME_f$ or $W_f = \Delta ME$, we didn't need to specify anything about work done by the force of friction or the force applied. Therefore, $W_{net} = \Delta KE$ is *always* true. And unlike some other alwayses♥, this always is always true. $W_{net} = \Delta KE$ is *always* true.

An aside: Now that we have these two equations $W_f = \Delta ME$ and $W_{net} = \Delta KE$, students often confuse the two of them. Students mistakenly interchange the ΔME and the ΔKE . Don't let this be you!!

♥ Find me a plural of always, I dare you!