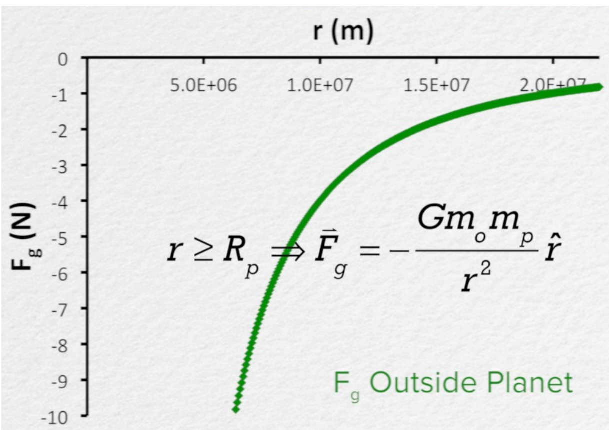
Flipping Physics Lecture Notes:

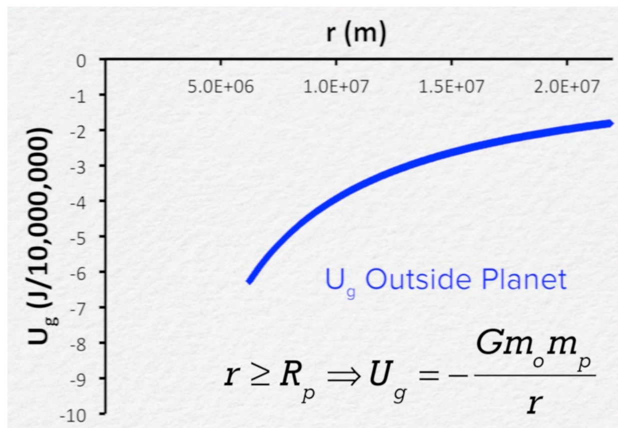
Force of Gravity and
Gravitational Potential Energy Functions from Zero to Infinity (but not beyond)

We already used Newton's Universal Law of Gravitation to determine the equation for Universal Gravitational Potential Energy. We did that by using the equation that relates a conservative force to the potential energy associated with that conservative force: $F_x = -\frac{dU}{dx}$. Remember, this equation is ***not*** on the AP Physics C: Mechanics equation sheet:

$$\vec{F}_g = -\frac{Gm_o m_p}{r^2}\hat{r} \ \& \ F_x = -\frac{dU}{dx} \Rightarrow U_g = -\frac{Gm_1 m_2}{r}$$

If we specify the two masses to be a planet and an object, we can draw the graphs of the force of gravity and gravitational potential energy as a function of position, r, where $r \geq R_p$





Notice the force of gravity is proportional to $\frac{1}{r^2}$, whereas the universal gravitational potential energy is proportional to $\frac{1}{r}$.

We also want to determine the universal gravitational potential energy *inside* the planet, where $r \leq R_p$. In order to determine this we first need the force of gravity *inside* the planet. In order to do that we need to make some assumptions:

1) The planet has a constant density, ρ.
    a. Yes, this is not actually true, but it is a good thought experiment.
2) We need to determine the force of gravity acting on an object that can move, without friction, through a tunnel we have drilled all the way through the center of the planet.
    a. And you thought assuming the planet had a constant density was a stretch…
3) In order to do this, we need to assume the planet is not rotating.
4) We need to assume the only mass of the planet that exerts a force of gravity on our object as $r < R_p$ is the mass of the planet that is inside a hypothetical sphere created by our variable r, the radius of the current location of the object. In other words, while all the mass of outside of r still causes a force of gravity on the object, due to symmetry and the fact that the force of gravity is proportional to the inverse square of the distance, all of the forces of gravitational attraction caused by the mass of the planet with $r \geq R_p$ cancel out. This is called Newton's Shell Theorem♥, in case you were curious.

♥ https://www.math.ksu.edu/~dbski/writings/shell.pdf

Here is the derivation of the force of gravitational attraction between an object and a planet as the object moves below the surface of the planet:

$$\rho = \frac{m_{total}}{\forall_{total}} = \frac{m_{in}}{\forall_{in}} \Rightarrow m_{in} = \rho \forall_{in} = \rho\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\rho r^3$$

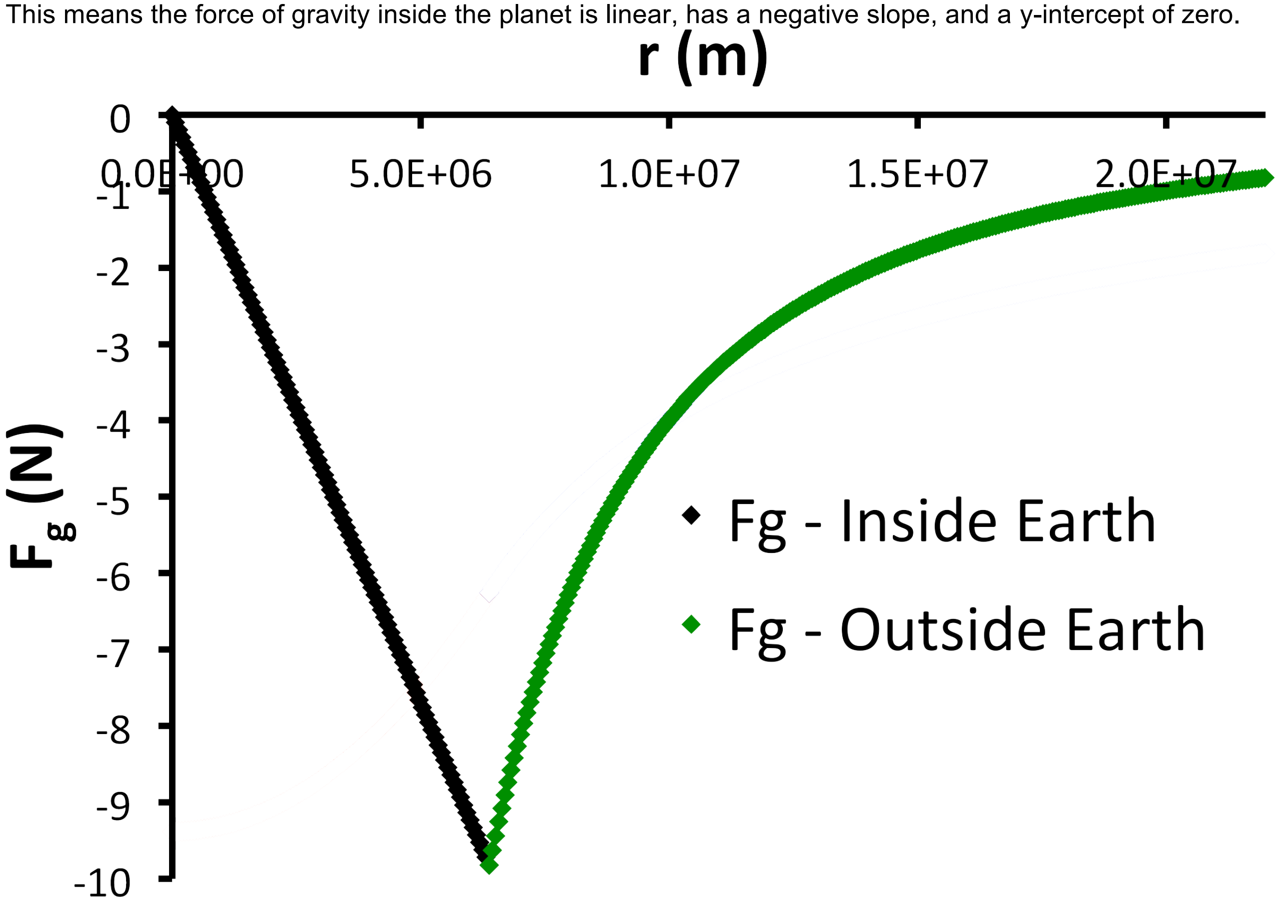
Note: m_{in} and $\forall_{in}$ refer to the mass and volume that are inside the sphere created by the variable r.

$$F_{g_{in}} = \frac{Gm_o m_{in}}{r^2} = \frac{Gm_o}{r^2}\left(\frac{4}{3}\pi\rho r^3\right) = \left(\frac{4\pi G m_o \rho}{3}\right) r$$

Notice $\frac{4\pi G m_o \rho}{3}$ are all constants.

In terms of vectors, the force of gravity is directed towards the center of the planet: $\vec{F}_{g_{in}} = -\left(\frac{4\pi G m_o \rho}{3}\right) r\ \hat{r}$

This means the force of gravity inside the planet is linear, has a negative slope, and a y-intercept of zero.



Now we can start working with energy and the conservative force equation:

$$F_x = -\frac{dU}{dx} \Rightarrow F_{g_{in}} = -\frac{dU_{g_{in}}}{dr} \Rightarrow dU_{g_{in}} = -F_{g_{in}} dr \Rightarrow \int dU_{g_{in}} = -\int F_{g_{in}} dr \Rightarrow U_{g_{in}} = -\int -\left(\frac{4\pi G m_o \rho}{3}\right) r\, dr$$

$$\Rightarrow U_{g_{in}} = \left(\frac{4\pi G m_o \rho}{3}\right)\int r\, dr = \left(\frac{4\pi G m_o \rho}{3}\right)\frac{r^2}{2} + C = \left(\frac{2\pi G m_o \rho}{3}\right) r^2 + C$$

To solve for C, we can use the fact that U_g has to have the same value for both expressions where $r = R_p$:

$$U_g\left(@ r = R_p\right) = \left(\frac{2\pi G m_o \rho}{3}\right) R_p^{\ 2} + C = -\frac{G m_o m_p}{R_p} \Rightarrow C = -\frac{G m_o m_p}{R_p} - \left(\frac{2\pi G m_o \rho}{3}\right) R_p^{\ 2}$$

$$\Rightarrow U_{g_{in}} = \left(\frac{2\pi G m_o \rho}{3}\right) r^2 - \left(\frac{G m_o m_p}{R_p} + \left(\frac{2\pi G m_o \rho}{3}\right) R_p^{\ 2}\right)$$

I know this looks confusing, however, the only variable in this whole equation is r, everything else is a constant. This equation is essentially $U_{g_{in}} = \# r^2 - \#$. It is a parabolic function shifted down on the y-axis.

The graph starts at a large negative number, is proportional to r^2 until r = R_{Earth} and then is proportional to 1/r:

For $r \le R_p$, $\vec{F}_{g_{in}} = -\left(\frac{4\pi G m_o \rho}{3}\right) r\ \hat{r}$ and for $r \ge R_p$, $\vec{F}_g = -\frac{G m_o m_p}{r^2}\hat{r}$

For $r \le R_p$, $U_{g_{in}} = \left(\frac{2\pi G m_o \rho}{3}\right) r^2 - \left(\frac{G m_o m_p}{R_p} + \left(\frac{2\pi G m_o \rho}{3}\right) R_p^{\ 2}\right)$ and for $r \ge R_p$, $U_g = -\frac{G m_o m_p}{r}$

A 1-kilogram mass with respect to the Earth has graphs which looks like this: