Flipping Physics Lecture Notes:

Moment of Inertia and Rotational Kinetic Energy Derivations

Our current equation for kinetic energy is $KE = \frac{1}{2}mv^2$.

According to our kinetic energy equation, when an object is rotating around its center of mass, but its center of mass is not moving, the object has zero velocity; therefore, the object has zero kinetic energy. Kinetic energy is the energy of motion. Hopefully you recognize that an object rotating around its center of mass is moving and therefore must have kinetic energy, it is just not described by the equation above. We need to look at the kinetic energy of all the individual pieces of the rotating object.

The total kinetic energy of all the individual pieces of the rotating object is: $KE_{total} = \sum_i KE_i$

This means the sum of all the kinetic energies of every piece that makes up the object. The letter "i" represents that the number goes from 1 to "i", the total number of pieces which make up the object. We can substitute in the equation for kinetic energy: $KE_{total} = \sum_i \frac{1}{2} m_i \left(v_i\right)^2$

The v_i represents the velocity of every piece which makes up the object. Notice this must be a tangential velocity, because the object is rotating and therefore every part of the object is moving in a circle.

Remember the equation that relates tangential velocity to angular velocity: $v_t = r\omega$

Therefore: $\Rightarrow KE_{total} = \sum_i \frac{1}{2} m_i \left(r_i \omega_i\right)^2 = \sum_i \frac{1}{2} m_i \left(r_i\right)^2 \left(\omega_i\right)^2$

Assuming the object is a *rigid object with shape*, the angular velocity of every piece of the object will be the same, therefore $\omega_i = \omega$

Notice the "r" in this equation is the distance each particle is from the axis of rotation, which is *not* the same for each piece.

We can isolate $\sum_i m_i \left(r_i\right)^2$ in the equation: $\Rightarrow KE_{total} = \frac{1}{2}\left(\sum_i m_i \left(r_i\right)^2\right)\omega^2$

We define $\sum_i m_i \left(r_i\right)^2$ as the *Moment of Inertia* of the object and identify it with the symbol, capital I:

$$\boxed{I = \sum_i m_i \left(r_i\right)^2}$$

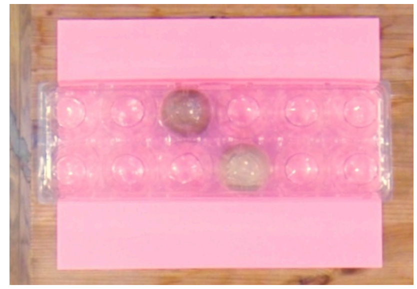
We can substitute the object's moment of inertia back into the total kinetic energy equation to get the total kinetic energy of a rotating object which is called *Rotational Kinetic Energy*: $\boxed{KE_{rotational} = \frac{1}{2} I \omega^2}$

The original kinetic energy then needs to be more specifically defined as *Translational Kinetic Energy*. In other words, the kinetic energy associated with the motion of the center of mass of the object moving from one point in space to another point in space: $KE_{translational} = \frac{1}{2}mv^2$

To help understand what moment of inertia is, notice the moment of inertia takes the place of the inertial mass in the kinetic energy equation. That is why I like to think of moment of inertia as "rotational mass".

Remember inertial mass is a measure of the tendency of an object to resist acceleration. The more mass something has, the more it resists acceleration. This means that moment of inertia or “rotational mass” is a measure of the tendency of an object to resist *angular* acceleration. The more moment of inertia or “rotational mass” something has, the more it resists *angular* acceleration.

Two Eggs in an Egg Carton: A Moment of Inertia or “Rotational Mass” Example:
For this example, we are going to assume the egg carton has a small enough mass relative to the mass of the two eggs to be negligible.

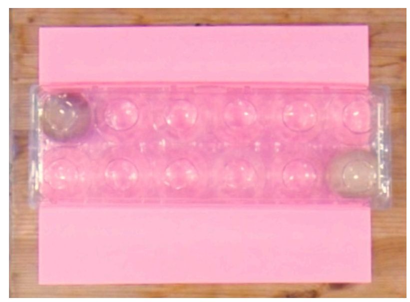


Place two eggs in an egg carton, both near the middle like this:
Because there are two objects in the system, the moment of inertia will be:

$$I=\sum_i m_i\left(r_i\right)^2 \Rightarrow I=m_1\left(r_1\right)^2+m_2\left(r_2\right)^2$$

Where m is the mass of each egg and r is the distance each egg is from the axis of rotation.

When we hold the egg carton in the middle and rotate it, it is relatively easy to rotate the system. In other words, because the eggs are close to the axis of rotation, the moment of inertia is low, and it is relatively easy to cause the eggs to angularly accelerate.



Now move the eggs so that they are on opposite ends of the egg carton like this:
The distance each egg is from the axis of rotation has been increased such that, now when we hold the egg carton in the middle and rotate it, it is more difficult to rotate the system. In other words, because we have increased the distance the eggs are from the axis of rotation, we have increased the moment of inertia or “rotational mass”, and therefore it is more difficult to cause the eggs to angularly accelerate.

Realize we have not changed the mass of the system; we have only changed the locations of the masses. Increasing the distance the eggs are from the axis of rotation increases the moment of inertia or “rotational mass” of the system which makes it more difficult to angularly accelerate; however, the inertial mass of the system remains the same.

Lastly, notice how “r”, the distance from the axis of rotation of each particle, is squared in the moment of inertia equation. This means the distance each particle is from the axis of rotation of the system has a much larger influence over the moment of inertia than the mass of each particle.