Flipping Physics Lecture Notes:

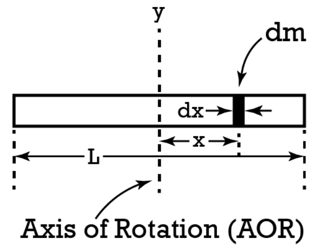
Using Integrals to Derive Rotational Inertia of a Long, Thin Rod with Demonstration

The equation for rotational inertia of a rigid object with shape is: $I=\int r^2\,dm$

We are going to determine the rotational inertia of a long, thin, uniform density rod. For that we need to use linear mass density, λ.

$$\lambda=\frac{M}{L}=\frac{dm}{dx}\Rightarrow dm=\lambda dx\Rightarrow dm=\frac{M}{L}dx$$

- M is the total mass of the rod
- L is the total length of the rod
- λ is the linear mass density of the rod, which is constant in this "uniform" rod.

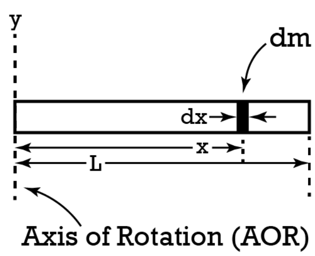


We can solve for the rotational inertia of the rod about its center of mass.

$$I_y=\int r^2\,dm=\int r^2\frac{M}{L}dx=\frac{m}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}x^2\,dx=\frac{M}{L}\left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$\Rightarrow I_y=\frac{M}{L}\left[\frac{\left(\frac{L}{2}\right)^3}{3}-\frac{\left(-\frac{L}{2}\right)^3}{3}\right]=\frac{M}{L}\left[\frac{L^3}{24}+\frac{L^3}{24}\right]=\frac{M}{L}\left[\frac{2L^3}{24}\right]=\boxed{\frac{1}{12}ML^2}$$

Moving the axis of rotation to the left end of the rod only changes the limits of the integral:



$$I_{end}=\frac{M}{L}\int_0^L x^2\,dx\Rightarrow\frac{M}{L}\left[\frac{x^3}{3}\right]_0^L=\frac{M}{L}\left[\frac{L^3}{3}-\frac{0^3}{3}\right]=\boxed{\frac{1}{3}ML^2}$$

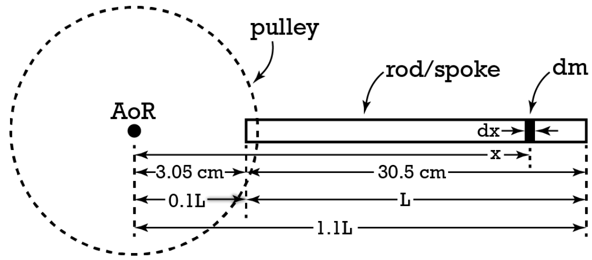
But now, we want to test our physics. In order to do so, we have the Rotational Inertia Demonstrator (RID) from Arbor Scientific. It is a pulley with three different pulley sizes and four long, thin, uniform density spokes which radiate from its center. Previously we measured the rotational inertia of the central pulley part which equals 0.00067 kg·m^2. Now we are going to use what we just learned to determine what the rotational inertia of the whole RID is:

$$I_{RID}=I_{pulley}+4I_{spoke}$$

Therefore, we need to determine the rotational inertia of a single spoke of the RID. To do this, we are going to use the same integral as before, and change the limits.

The length of one spoke is 30.5 cm and the end of the spoke starts 3.05 cm from the axis of rotation. Therefore the initial point is 0.1L: $x_i=\frac{3.05}{30.5}L=0.1L$

And the final point is 1.1 L: $x_f=\frac{3.05+30.5}{30.5}L=1.1L$



$$I_{spoke}=\frac{M}{L}\int_{0.1L}^{1.1L}x^2\,dx \Rightarrow \frac{M}{L}\left[\frac{x^3}{3}\right]_{0.1L}^{1.1L}=\frac{M}{L}\left[\frac{(1.1L)^3}{3}-\frac{(0.1L)^3}{3}\right]=\frac{M}{3L}\left[1.331L^3-0.001L^3\right]$$

$$\Rightarrow I_{spoke}=\frac{M}{3L}\left[1.33L^3\right]=0.44\bar{3}ML^2\approx\boxed{0.443ML^2}$$

Knowns: $M=0.0742kg;\ L=0.305m$

$$\Rightarrow I_{RID}=I_{pulley}+4\left(0.44\bar{3}ML^2\right)=0.00067+(4)(0.44\bar{3})(0.0742)(0.305)^2=0.01292685$$

$$\Rightarrow I_{RID}\approx\boxed{0.0129kg\cdot m^2}$$

And we can test this. Previously we solved for the rotational inertia of an object in terms of the pulley radius, angular acceleration, and the force of tension acting on the pulley. See: “Graphing the Rotational Inertia of an Irregular Shape”. https://www.flippingphysics.com/rotational-inertia-irregular-shape.html

$$\Delta t=133frames\times\frac{1\sec}{60frames}=2.21\bar{6}\sec;\ \Delta\theta=2rev\times\frac{2\pi rad}{1rev}=4\pi rad;\ \omega_i=0$$

$$\alpha=\frac{2\Delta\theta}{\Delta t^2}=\frac{(2)(4\pi)}{(2.221\bar{6})^2}=5.11492\frac{rad}{s^2};\ R_{pulley}=0.0286m;\ F_{T_{avg}}=2.452N$$

$$I=\frac{R_{pulley}F_T}{\alpha}=\frac{(0.0286)(2.452)}{5.11492}=0.013710\approx0.0137kg\cdot m^2$$

$$E_r=\frac{O-A}{A}\times100=\frac{0.13710-0.01292685}{0.01292685}\times100=6.06075\approx6.06\%$$