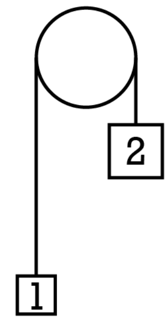
Flipping Physics Lecture Notes:

2 Masses on a Pulley
Conservation of Energy Demonstration

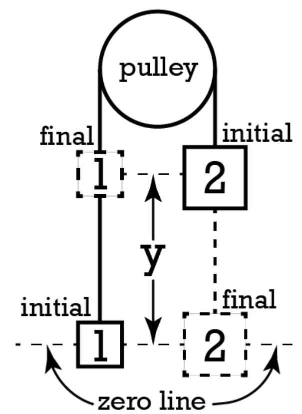


Example: Mass 1 and mass 2 hang from either side of a frictionless pulley with rotational inertia, I, and radius, R. What is the angular acceleration of the pulley?

Assume the system starts at rest and $m_2 > m_1$.

Knowns: $m_1, m_2, I, R, \alpha = ?$

There is neither a force applied or a force of friction which is adding or removing energy from the system, therefore, mechanical energy is conserved. Define the initial point as where the objects are when they are at rest and the final point as where the objects are after mass 1 has moved up a distance y and mass 2 has moved down the same distance y. Set the zero line at the initial point.



$$ME_i = ME_f \Rightarrow PE_{g_{2i}} = PE_{g_{1f}} + KE_{T_{1f}} + KE_{T_{2f}} + KE_{R_{pf}}$$

Initially the only type of mechanical energy is gravitational potential energy of mass 2. Finally, mass 1 has gravitational potential energy, both masses have translational kinetic energy, and the pulley has rotational kinetic energy. Note: The gravitational potential energy of the pulley is the same initially and finally, therefore it would have the same value on either side of the equal sign and would cancel out. That is why we have not included it.

Substituting in mechanical energy equations:

$$\Rightarrow m_2 g h_{2i} = m_1 g h_{1f} + \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 + \frac{1}{2} I \omega_{pf}^2$$

$$h_{2i} = h_{1f} = y \Rightarrow m_2 g y = m_1 g y + \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 + \frac{1}{2} I \omega_{pf}^2$$

Because they are attached by the string, the velocities of both masses are the same as the tangential velocity of the rim of the pulley: $v_t = r\omega \Rightarrow v_t = v_{1f} = v_{2f} = R\omega_{pf}$ (and multiply through by 2)

$$\Rightarrow 2m_2 g y = 2m_1 g y + m_1 \left(R\omega_{pf}\right)^2 + m_2 \left(R\omega_{pf}\right)^2 + I\omega_{pf}^2$$

$$\Rightarrow 2m_2 g y = 2m_1 g y + m_1 R^2 \omega_{pf}^2 + m_2 R^2 \omega_{pf}^2 + I\omega_{pf}^2$$

$$\Rightarrow 2m_2 g y - 2m_1 g y = \omega_{pf}^2 \left(m_1 R^2 + m_2 R^2 + I\right) \Rightarrow \omega_{pf}^2 = \frac{2m_2 g y - 2m_1 g y}{m_1 R^2 + m_2 R^2 + I}$$

Again, because they are attached by the string, both masses go through the same linear displacement, y, which is the same as the arc length traveled by the rim of the pulley: $s = r\Delta\theta \Rightarrow y = R\Delta\theta \Rightarrow \Delta\theta = \frac{y}{R}$

And the torque on the pulley will be constant so the pulley will experience constant angular acceleration, so we can use one of the uniformly angularly accelerated motion equations:

$$\omega_f^{\,2} = \omega_i^{\,2} + 2\alpha\Delta\theta = 2\alpha\Delta\theta \Rightarrow \alpha = \frac{\omega_f^{\,2}}{2\Delta\theta} = \frac{R\omega_f^{\,2}}{2y}$$

And then we can substitute in the equation we derived for the square of the final angular velocity of the pulley.

$$\Rightarrow \alpha = \left(\frac{R}{2y}\right)\left(\frac{2m_2gy - 2m_1gy}{m_1R^2 + m_2R^2 + I}\right) = \boxed{\frac{Rg\left(m_2 - m_1\right)}{R^2\left(m_1 + m_2\right) + I}}$$

This matches our answer from when we previously did this problem using Newton's Second Law, both the translation and rotational forms: https://www.flippingphysics.com/2-mass-pulley-torque.html