Any object flying through the vacuum you can breathe in both the x and y directions is in projectile motion. When solving a projectile motion problem you need to separate the x and y direction variables.

x direction	y direction
$a_x = 0$	Free-Fall
Constant Velocity	$a_y = -g = -9.81 \dfrac{m}{s^2}$
$v_x = \dfrac{\Delta x}{\Delta t}$ (need to know 2 variables)	Uniformly Accelerated Motion (need to know 3 variables)
Δt is the same in both directions because it is a *scalar* and has magnitude only (no direction).	

List what you know in both the x and y directions and solve for Δt in one direction and then use it in the other direction.

The only equation in the x direction is $v_x = \dfrac{\Delta x}{\Delta t}$, therefore there are 3 variables in the x direction: v_x, Δx &

Δt. Therefore, you need to know 2 variables in the x direction to find the other 1.

In the y direction we have Uniformly Accelerated Motion, the equations for which are:

There are 5 variables in the UAM equations: $v_f, v_i, a, \Delta t, \& \Delta x$

There are 4 equations.
If you know 3 of the variables
you can find the other 2.
Which leaves you with 1 …

$$v_f = v_i + a\Delta t$$

$$v_f^{\,2} = v_i^{\,2} + 2a\Delta x$$

$$\Delta x = v_i \Delta t + \frac{1}{2}a\Delta t^2$$

$$\Delta x = \frac{1}{2}\left(v_f + v_i\right)\Delta t$$

Example Problem: While in a car moving at 10.0 miles per hour, mr.p drops a ball from a height of 0.70 m above the top of a bucket. Part 1) How far in front of the bucket should he drop the ball such that the ball will land in the bucket?

This is a projectile motion problem, so we should list our know variables in the x and y directions:

x-direction: $\Delta x = ?, v_x = 10.0\dfrac{mi}{hr}\times\dfrac{1hr}{3600\sec}\times\dfrac{1609m}{1mi} = 4.469\overline{4}\dfrac{m}{s}$

y-direction: $\Delta y = -0.70m, a_y = -9.81\dfrac{m}{s^2}, v_{iy} = 0$

Because we know three variables in the y direction, we should start there to find the change in time.

$$\Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \Rightarrow -0.7 = (0)\Delta t + \frac{1}{2}(-9.81)\Delta t^2 \Rightarrow (2)(-0.7) = -9.81\Delta t^2$$

$$\Rightarrow \Delta t^2 = \frac{(2)(-0.7)}{-9.81} \Rightarrow \Delta t = \sqrt{\frac{(2)(-0.7)}{-9.81}} = 0.377772\sec$$

Because change in time is a scalar and therefore independent of direction, we can use it in the x-direction:

$$v_x = \frac{\Delta x}{\Delta t} \Rightarrow \Delta x = v_x\Delta t = (4.469\overline{4})(0.377772) = 1.68843 \approx \boxed{1.7m}$$

In the video I said, "If you decrease the speed of the car by half a mile per hour, you decrease the displacement in the x-direction of the ball by more than 8 cm." Here is the mathematical proof:

Let's start by decreasing the car by half a mile per hour, this makes the initial velocity in the x-direction 9.5 mph.

Known variables:

x-direction: $\Delta x = ?, v_x = 9.5\dfrac{mi}{hr}\times\dfrac{1hr}{3600\sec}\times\dfrac{1609m}{1mi} = 4.24597\overline{2}\dfrac{m}{s}$

y-direction: $\Delta y = -0.70m, a_y = -9.81\dfrac{m}{s^2}, v_{iy} = 0$

The y-direction information doesn't change at all so we still get 0.377772 seconds for the change in time.

$$\Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \Rightarrow -0.7 = (0)\Delta t + \frac{1}{2}(-9.81)\Delta t^2 \Rightarrow (2)(-0.7) = -9.81\Delta t^2$$

$$\Rightarrow \Delta t^2 = \frac{(2)(-0.7)}{-9.81} \Rightarrow \Delta t = \sqrt{\frac{(2)(-0.7)}{-9.81}} = 0.377772\sec \text{ (see, it didn't change! ☺)}$$

Because change in time is a scalar and therefore independent of direction, we can use it in the x-direction:

$$v_x = \frac{\Delta x}{\Delta t} \Rightarrow \Delta x = v_x\Delta t = (4.24597\overline{2})(0.377772) = 1.60401m$$

The difference in x displacement then is:

$$1.68843 - 1.60401 = 0.084421m\times\frac{100cm}{1m} \approx \boxed{8.4cm} \text{ (which is greater than 8 cm.)}$$

Flipping Physics Lecture Notes:
An Introductory Projectile Motion Problem
with an Initial Horizontal Velocity
(Part 2 of 2)

Example Problem: While in a car moving at 10.0 miles per hour, mr.p drops a ball from a height of 0.70 m above the top of a bucket. Part 1) How far in front of the bucket should he drop the ball such that the ball will land in the bucket? *(Yes, I left the solution to part 1 in here. It just seemed more logical to me that way.)*

This is a projectile motion problem, so we should list our know variables in the x and y directions:

x-direction: $\Delta x = ?, \; v_x = 10.0 \dfrac{mi}{hr} \times \dfrac{1hr}{3600\sec} \times \dfrac{1609m}{1mi} = 4.469\overline{4}\dfrac{m}{s}$

y-direction: $\Delta y = -0.70m, \; a_y = -9.81\dfrac{m}{s^2}, \; v_{iy} = 0$

Because we know three variables in the y direction, we should start there to find the change in time.

$$\Delta y = v_{iy}\Delta t + \dfrac{1}{2}a_y\Delta t^2 \Rightarrow -0.7 = (0)\Delta t + \dfrac{1}{2}(-9.81)\Delta t^2 \Rightarrow (2)(-0.7) = -9.81\Delta t^2$$

$$\Rightarrow \Delta t^2 = \dfrac{(2)(-0.7)}{-9.81} \Rightarrow \Delta t = \sqrt{\dfrac{(2)(-0.7)}{-9.81}} = 0.377772\sec$$

Because change in time is a scalar and therefore independent of direction, we can use it in the x-direction:

$$v_x = \dfrac{\Delta x}{\Delta t} \Rightarrow \Delta x = v_x\Delta t = (4.469\overline{4})(0.377772) = 1.68843 \approx \boxed{1.7m}$$

Part 2) what is the final velocity of the ball right before it enters the bucket? In the y-direction:

$$v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y = 0^2 + (2)(-9.81)(-0.7) \Rightarrow v_{fy} = \sqrt{(2)(-9.81)(-0.7)} = \pm3.70594 = -3.70594\dfrac{m}{s}$$
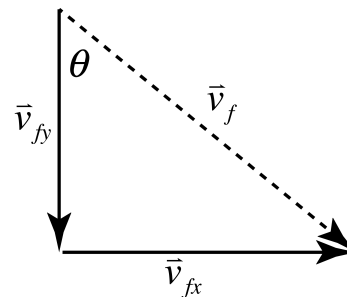
In the x-direction the ball is moving with a constant velocity and therefore

$v_x = 4.469\overline{4}\dfrac{m}{s}$. Now, to find the magnitude of the final velocity we can use

the Pythagorean theorem:

$$a^2 + b^2 = c^2 \Rightarrow v_f^2 = v_{fx}^2 + v_{fy}^2$$

$$\Rightarrow v_f = \sqrt{v_{fx}^2 + v_{fy}^2} = \sqrt{(4.469\overline{4})^2 + (-3.70594)^2} = \pm5.80602\dfrac{m}{s}$$

And now we need to find the direction:

$$\sin\theta = \dfrac{O}{H} = \dfrac{v_{fx}}{v_f} \Rightarrow \theta = \sin^{-1}\left(\dfrac{v_{fx}}{v_f}\right) = \sin^{-1}\left(\dfrac{4.469\overline{4}}{5.80602}\right) = 50.3354°$$

Therefore the final velocity is: $\boxed{v_f \approx 5.8\dfrac{m}{s} @ (5.0\times10^1)° \text{ in front of the -y axis.}}$

Possible useful definitions:
- Cheeky (adjective): impudent or irreverent, typically in an endearing or amusing way.
- Gormless (adjective): lacking sense or initiative; foolish.
- Nimwit (noun): a stupid or silly person.
- Ill-fated (adjective): destined to fail or have bad luck.
- Buffoon (noun): a ridiculous but amusing person; a clown.

Flipping Physics Lecture Notes:
A Brief Look at the Force of Drag using
Numerical Modeling (or The Euler Method)

We did this example problem in the last lesson. *Example Problem:* While in a car moving at 10.0 miles per hour, mr.p drops a ball from a height of 0.70 m above the top of a bucket. How far in front of the bucket should he drop the ball such that the ball will land in the bucket?

Using the assumption that their was no air resistance, we used the concept of projectile motion to go from our initial known values to determine the displacement in the x-direction to be 1.68843 meters. In the video I stated that air resistance decreased the displacement in the x-direction by "less than 1 cm."

Let's talk about air resistance. When we add air resistance the object is no longer in projectile motion. In other words, the acceleration in the y-direction is *not* a constant -9.81 m/s^2 and the velocity in the x-direction is *not* constant. In order to figure out the acceleration of the ball in both the x and y directions, we need to draw a free body diagram of the forces acting on the ball while it is in the *air*.

The ball is moving to the right, so there will be a force of drag opposite that motion to the left. The ball is also moving down, so there will be a force of drag opposite that motion and down. These are actually the components of the net force of drag, however, it is easier to look at the drag in this case in terms of its components.
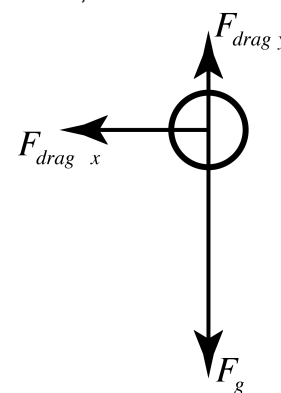
A typical equation for the force of drag is $F_{drag} = \frac{1}{2}\rho v^2 DA$, where ρ is *the density of the medium* through which the object is moving, v is the velocity of the object, D is called the Drag Coefficient and A is the cross sectional area of the object normal to the direction of travel. The density of air is not constant and is dependent on temperature and pressure, so, 1.275 kg/m^3 is a good approximation for today.[1] The cross section of a sphere in any direction is a circle. The ball that I used was a lacrosse ball for which there are clear specifications[2] from which you can determine that the radius of the ball is approximately 0.031835 m and the mass of the ball is approximately 0.14529 kg.

The drag coefficient of an object is defined by NASA as "a number which aerodynamicists use to model all of the complex dependencies of drag on shape, inclination, and some flow conditions."[3] Basically, it is an experimentally determined number that helps determine the drag on an object and changes depending on the shape of the object, the type of fluid flow around the object and the speed of the object. The drag coefficient of a baseball, according to NASA, is approximately 0.3, however, this isn't a baseball, so they alsp approximate a smooth sphere (like a lacrosse ball) to have a drag coefficient of around 0.5.[4] We will use 0.5 for the drag coefficient of our lacrosse ball.

A very important thing to notice about the drag force is that it is proportional to the square of the velocity. This means that as the velocity changes, the drag force changes, which changes the acceleration, which changes the velocity, which changes the drag force, which changes the acceleration, which changes the velocity … you get my point. This is certainly *not* uniformly accelerated motion, UAM, and the way we have to deal with it is by using Numerical Modeling other wise known as the Euler Method.

Leonhard Euler (1707-1783) devised this method of solving problems among other things, including finding Euler's number or the e number, 2.71828. Numerical Modeling or Euler's Method uses the idea that we can approximate the acceleration to be constant for a very short time interval and then we can use the UAM equations for that short time interval and come up with a good approximation of the motion. The shorter the time interval, the better the approximation.

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[1] http://chemistry.about.com/od/gases/f/What-Is-The-Density-Of-Air-At-Stp.htm
[2] http://www.uslacrosse.org/portals/1/documents/pdf/about-the-sport/nocsae-ball-standards.pdf
[3] http://www.grc.nasa.gov/WWW/k-12/airplane/shaped.html
[4] https://www.grc.nasa.gov/www/k-12/airplane/balldrag.html

Let's sum the forces in the x-direction and assume $v_{ix} = 10.0 \dfrac{mi}{hr} \times \dfrac{1hr}{3600\sec} \times \dfrac{1609m}{1mi} = 4.469\overline{4}\,\dfrac{m}{s}$

$$\sum F_x = -F_{drag\,x} = ma_x \Rightarrow -\frac{1}{2}\rho_{air}v_{ix}^2 DA = ma_x \Rightarrow a_x = -\frac{\rho_{air}v_{ix}^2 D(\pi r^2)}{2m}$$

$$\Rightarrow a_x = -\frac{(1.275)(4.469\overline{4})^2(0.5)\left(\pi(0.031835)^2\right)}{2(0.14529)} = -0.13953\,\frac{m}{s^2}$$

Now that we know the acceleration in the x-direction, we can approximate that acceleration to be constant for $1/100^{th}$ of a second and determine v_{fx} using a UAM equation.

$$v_{fx} = v_{ix} + a_x\Delta t = (4.469\overline{4}) + (-0.13953)(0.01) = 4.46805\,\frac{m}{s}$$

To find the position in the x-direction after $1/100^{th}$ of a second, we again use a UAM equation.

$$\Delta x = x_f - x_i = v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 \Rightarrow x_f = x_i + v_{ix}\Delta t + \frac{1}{2}a_x\Delta t^2 = (0) + (4.469\overline{4})(0.01) + \frac{1}{2}(-0.13953)(0.01)^2$$

$\Rightarrow x_f = 0.04469m$ And these become the initial velocity and initial position for the next $1/100^{th}$ of a second and we can determine the acceleration, then the final velocity, then the final position and this is why we have spreadsheets. We can Harness the Power of Excel!!

Let's look at the y-direction.

$$\sum F_y = F_{drag\,y} - F_g = \frac{1}{2}\rho_{air}v_{iy}^2 DA - mg = ma_y \Rightarrow \frac{1}{2}\rho(0)^2 DA - mg = ma_y \Rightarrow -mg = ma_y \Rightarrow a_y = -g = -9.81\,\frac{m}{s^2}$$

So at the very, very beginning, when the initial velocity in the y-direction equals zero, then the acceleration in the y-direction is -9.81 m/s^2, after that, the acceleration will begin to move toward zero.

To find the final velocity in the y-direction after $1/100^{th}$ of a second, we again use a UAM equation.

$$v_{fy} = v_{iy} + a_y\Delta t = (0) + (-9.81)(0.01) = -0.0981\,\frac{m}{s}$$

To find the position in the x-direction after $1/100^{th}$ of a second, we again use a UAM equation.

$$\Delta y = y_f - y_i = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \Rightarrow y_f = y_i + v_{iy}\Delta t + \frac{1}{2}a_x\Delta t^2 = (0) + (0)(0.01) + \frac{1}{2}(-9.81)(0.01)^2 = -4.905 \times 10^{-4}m$$

Now we move on to the next $1/100^{th}$ of a second:

$$\sum F_y = F_{drag\,y} - F_g = \frac{1}{2}\rho v_{iy}^2 DA - mg = ma_y \Rightarrow a_y = \frac{\rho v_{iy}^2 D\pi r^2}{2m} - g$$

$$\Rightarrow a_y = \frac{(1.275)(-0.0981)^2(0.5)\left(\pi(0.031835)^2\right)}{2(0.14529)} - 9.81 = -9.80993\,\frac{m}{s^2}$$

And again we Harness the Power of Excel!! And you can see that, when we include air resistance, the ball will move 1.67984 and not 1.68843 meters in the x-direction. Which is a difference of:

$$1.68843 - 1.67984 = 0.00859m \times \frac{1000mm}{1m} = 8.59mm \approx \boxed{9mm}\text{, which is less than 1 cm.}$$

The Excel file I refer to can be found @ www.flippingphysics.com/euler-method.html

First off, the Nerd-A-Pult can be purchased at www.marshmallowcatapults.com

Example Problem: A ball is launched from the Nerd-A-Pult with an initial speed of 3.25 m/s at an angle of 61.7° above the horizontal. If the basket is 93 cm from the ball horizontally, where should the basket be placed vertically relative to the ball so the ball lands in the basket?

Before we start listing what we know in the x and y directions, we should split the initial velocity in to it's components.

$$\sin\theta = \frac{O}{H} \Rightarrow \sin\theta_i = \frac{v_{iy}}{v_i} \Rightarrow v_{iy} = v_i \sin\theta_i = (3.25)\sin(61.7°) = 2.86155\frac{m}{s}$$

$$\cos\theta = \frac{A}{H} \Rightarrow \cos\theta_i = \frac{v_{ix}}{v_i} \Rightarrow v_{ix} = v_i \cos\theta_i = (3.25)\cos(61.7) = 1.54079\frac{m}{s}$$
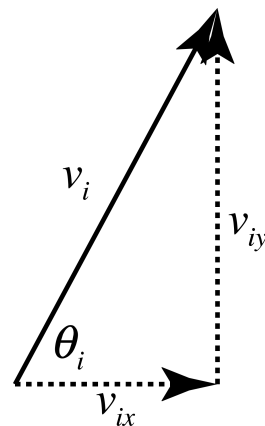
Now we can list our x & y direction knowns.

x-direction: $v_{ix} = 1.54079\frac{m}{s}$, $\Delta x = 93cm \times \frac{1m}{100cm} = 0.93m$

y-direction: $v_{iy} = 2.86155\frac{m}{s}$, $a_y = -9.81\frac{m}{s^2}$, $\Delta y = ?$

We know two variables in both the x and y directions so we should start by finding the change in time in the x-direction and then use that Δt in the y-direction. We can't start with the y-direction because we need to know 3 variable in the y-direction and only 2 in the x-direction.

x-direction: Remember the velocity initial in the x-direction is the same as the velocity in the x-direction because all objects in projectile motion have a constant velocity in the x-direction.

$$v_{ix} = v_x = \frac{\Delta x}{\Delta t} \Rightarrow v_x \Delta t = \Delta x \Rightarrow \Delta t = \frac{\Delta x}{v_x} = \frac{0.93}{1.54079} = 0.603588\sec \,^{♥}$$

y-direction: $\Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 = (2.86155)(0.603588) + \frac{1}{2}(-9.81)(0.603588)^2 = -0.059785m$

$$\Rightarrow \Delta y = -0.059785m \times \frac{100cm}{1m} = -5.9785cm \approx \boxed{-6.0cm}$$

♥ For those of you who plug this into your calculator and get $\Delta t = \frac{0.93}{1.54079} = 0.6035864719 \approx 0.603587\sec$, please realize that I usually use the answer button on my calculator instead of typing in the numbers. Therefore I didn't actually use 1.54079 for my v_x, I used 1.540786679 which gives you $\Delta t = \frac{0.93}{1.540786679} = 0.6036877727 \approx 0.603588\sec$. After rounding at the end of the problem, it doesn't matter, however, I knew there would be some people who would question my math. ☺

There are two things to measure, the initial speed and the initial angle. Let's start with the initial angle. I measured the hypotenuse of the triangle as 25.5 cm and took two measurement do determine the y side of the triangle: $y = 16.5 - 4.4 = 12.1cm$. Now we can find theta: $\sin\phi = \dfrac{O}{H} = \dfrac{y}{H}$

$\Rightarrow \phi = \sin^{-1}\left(\dfrac{y}{H}\right) = \sin^{-1}\left(\dfrac{12.1}{25.5}\right) = 28.327°$

Now we need to look at the initial velocity angle with the horizontal. Notice that the horizontal line for the initial velocity creates a smaller similar triangle with the original larger triangle.
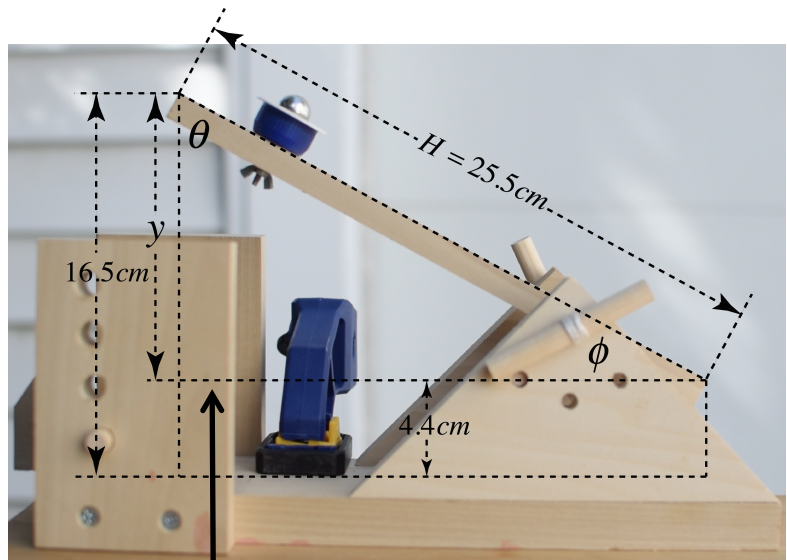
This shows that, because the interior angles of a triangle add up to 180°, the initial launch angle with the horizontal is θ. And we can determine that angle:

$180° = 90° + \phi + \theta \Rightarrow \theta = 180° - 90° - \phi = 90° - \phi$

$180° = 90° + \phi + \theta_i \Rightarrow \theta_i = 180° - 90° - \phi = 90° - \phi$

$\Rightarrow \theta_i = \theta = 90° - \phi = 90° - 28.327° = 61.673° \approx 61.7°$
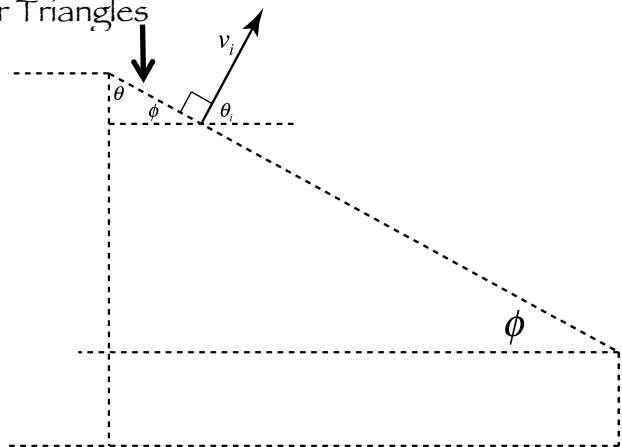
Now we need to determine the initial speed. For this I filmed several launches at 240 frames per second and measured the distance traveled by the ball in one frame. There were 5 that traveled 1.4 cm and 4 that traveled 1.3 cm for an average of: $distance_{avg} = \dfrac{(1.4 \times 5) + (1.3 \times 4)}{9} = 1.3\overline{5}cm$

Because there were 240 frames per second, that means that each frame lasts for 1/240th of a second.

$\left(240\dfrac{frames}{second}\right)^{-1} = \dfrac{1}{240}\dfrac{seconds}{frame}$ & then using the equation for average speed I determined the average initial speed:

$speed_{avg} = \dfrac{distance_{avg}}{time_{avg}} = \dfrac{1.3\overline{5}cm}{1/240\,sec} = 325.\overline{3}\dfrac{cm}{s} \times \dfrac{1m}{100cm} = 3.25\overline{3}\dfrac{m}{s} \approx 3.25\dfrac{m}{s}$

Therefore, with 3 significant figures: $\boxed{v_i = 3.25\dfrac{m}{s}\,@\,61.7° \text{ above the horizontal.}}$

It didn't occur to me until after I made the first video that I really only should have had 2 significant digits on the initial speed measurements because the original distance measurements only had 2 sig figs, oops.

Also, the change in time in the "air" in the Nerd-A-Pult video is about 1-2 frames shorter than it should be, I think there may be some error in the measurement of the initial launch angle because the wooden beam holding ball bent slightly on contact, which is something I was unable to measure.

The Nerd-A-Pult can be purchased at www.marshmallowcatapults.com

Example Problem: A ball is launched from the Nerd-A-Pult with an initial speed of 3.25 m/s at an angle of 61.7° above the horizontal. If the basket is 8.7 cm above the ball vertically, where should the basket be placed horizontally relative to the ball so the ball lands in the basket?
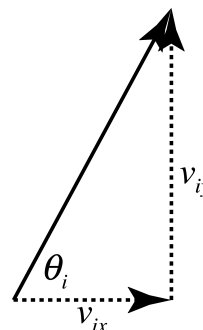
The initial velocity is the same as before, so we end up with the same components in the x and y directions.

$$\sin\theta = \frac{O}{H} \Rightarrow \sin\theta_i = \frac{v_{iy}}{v_i} \Rightarrow v_{iy} = v_i \sin\theta_i = (3.25)\sin(61.7°) = 2.86155\frac{m}{s}$$

$$\cos\theta = \frac{A}{H} \Rightarrow \cos\theta_i = \frac{v_{ix}}{v_i} \Rightarrow v_{ix} = v_i \cos\theta_i = (3.25)\cos(61.7) = 1.54079\frac{m}{s}$$

Now we can list our x & y direction knowns.

x-direction: $v_{ix} = v_x = 1.54079\frac{m}{s}$, $\Delta x = ?$

y-direction: $\Delta y = 8.7cm \times \frac{1m}{100cm} = 0.087m$; $a_y = -9.81\frac{m}{s^2}$ (remember $g_{Earth} = +9.81\frac{m}{s^2}$)

We know 1 variable in x-direction and 3 variables in the y-direction, so we should start with the y-direction and find the change in time there. We would need to know 2 variables to start in the x-direction.

y-direction: We can solve for Δt in two ways (1) directly, using the quadratic formula, or (2) we can solve for v_{fy} first and then Δt. For completeness sake, I will show both. FYI: In my experience, more students make mistakes when trying to use the quadratic formula then when solving for v_{fy} first.

$$(1)\ \Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \Rightarrow 0.087 = 2.86155\Delta t + \frac{1}{2}(-9.81)\Delta t^2 \Rightarrow -4.905\Delta t^2 + 2.86155\Delta t - 0.087 = 0$$

$$\Delta t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2.86155 \pm \sqrt{2.86155^2 - 4(-4.905)(-0.087)}}{(2)(-4.905)} = \frac{-2.86155 \pm 2.54588}{-9.81}$$

$$\Delta t = 0.291697 \pm 0.259519 = 0.551216\ or\ 0.0321784\sec \Rightarrow 0.55122\sec$$

(The time clearly isn't as small as ~3/100^{ths} of a second & for the quadratic formula $a\Delta t^2 + b\Delta t + c = 0$)

$$(2)\ v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y \Rightarrow v_{fy} = \sqrt{v_{iy}^2 + 2a_y\Delta y} = \sqrt{(2.86155)^2 + (2)(-9.81)(0.087)} = \pm 2.54588\frac{m}{s} = -2.54588\frac{m}{s}$$

Please be smarter than your calculator and, anytime you take the square root, remember that the solution can be positive or negative. We know v_{fy} is negative, because we know the ball is going down.

$$v_{fy} = v_{iy} + a_y\Delta t \Rightarrow v_{fy} - v_{iy} = a_y\Delta t \Rightarrow \Delta t = \frac{v_{fy} - v_{iy}}{a_y} = \frac{(-2.54588 - 2.86155)}{-9.81} = 0.55122\sec$$

Notice that there is no confusion over which time to pick when you don't use the quadratic formula.

Now that we know the change in time, we can switch to the x-direction:

$$v_x = \frac{\Delta x}{\Delta t} \Rightarrow \Delta x = v_x\Delta t = (1.54079)(0.55122) = 0.849314m \times \frac{100cm}{1m} = 84.9314cm \approx \boxed{85cm}$$

Flipping Physics Lecture Notes:
Understanding the Range Equation of Projectile Motion

The range of an object in projectile motion means something very specific. It is the displacement in the x direction of an object whose displacement in the y direction is zero.

Students often get confused by the statement "displacement in the y direction is zero" or Δy = 0. This does *not* mean that the object does not move up or down, it simply means that it ends at the same height it started as: Δy = y_f – y_i = 0
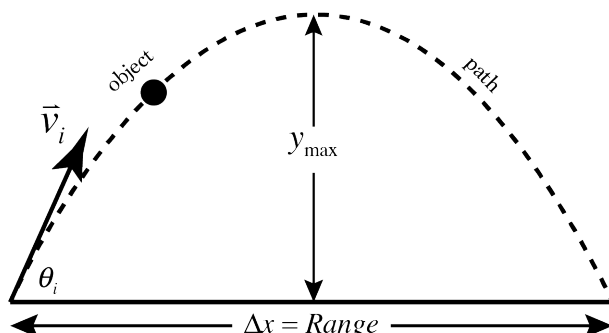
The Range Equation is $\boxed{R = \dfrac{v_i^2 \sin(2\theta_i)}{g}}$ & the variables in the range equation are:

- $\Delta x = Range = R$ (in other words, "R", stands for Range. Neds to be in meters.)
- $v_i \Rightarrow \|v_i\|$ (the magnitude of the initial velocity. Needs to be in meters per second.)
- $\theta_i \Rightarrow$ (the initial angle or launch angle. Usually in degrees & has to match your calculator mode.)
- $g_{Earth} = +9.81\dfrac{m}{s^2}$ (remember g is a *positive* number)

We can determine the angle that will give the largest range with the same magnitude initial velocity by remembering that the maximum value for the sine of any angle is 1:
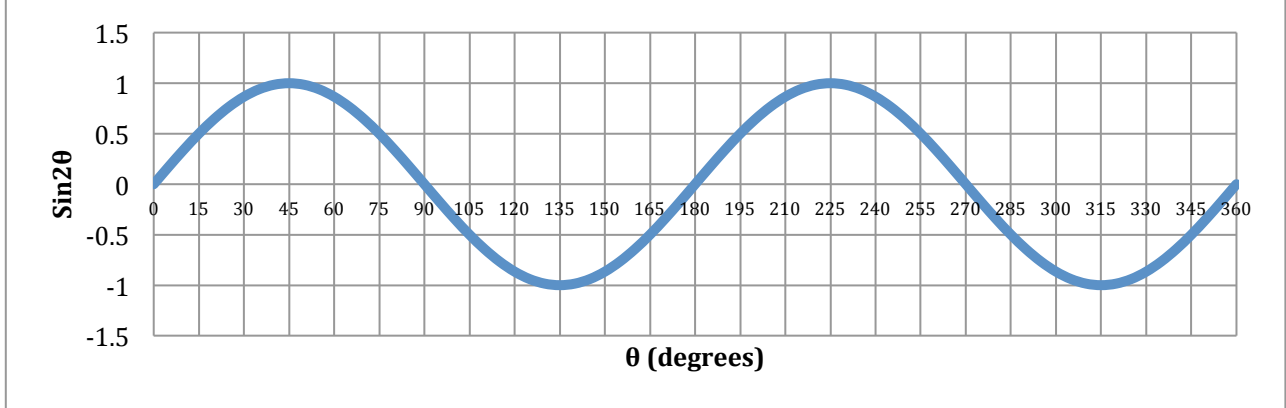
$$\text{Max Value} = 1 = \sin(2\theta_i) \Rightarrow 2\theta_i = \sin^{-1}(1) = 90° \Rightarrow \theta_i = \frac{90°}{2} = 45°$$

Therefore the maximum range is when θ_i is 45°

Also, remember the shape of the sin(θ) curve:
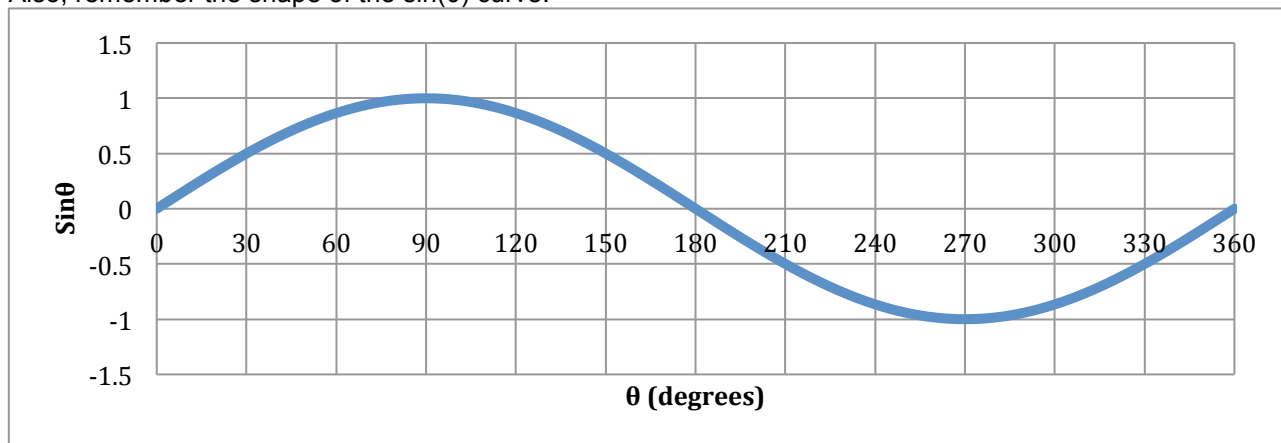
However, because this is sin(2θ), the curve is a bit different:

This means that $\sin(2\theta_i) = \sin(2(90 - \theta_i))$. Which means that there are two different angles that will have the same range. For example, $\theta_{1i} = 30° \, \& \, \theta_{2i} = (90 - \theta_{1i}) = (90 - 30) = 60°$, because both have the same value for $\sin(2\theta_i)$:

You can also see that $\sin(2\theta_i) = \sin(2(90 - \theta_i))$ or $\sin(2\theta_i) = \sin(2(90 - \theta_i))$ when I add a horizontal line to the above graph. And you can see that the y-value on the graph is the same for both 30° and 60°.
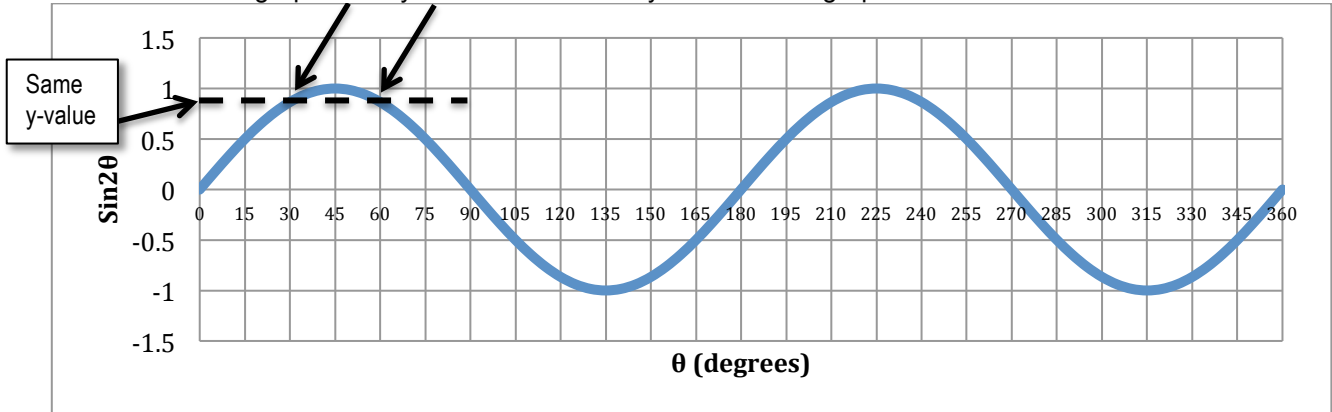
Just so you know, θ_{1i} & θ_{2i} are complimentary angles because they add up to 90°. So two launch angles that are complimentary will result in the same range.

Now, θ_{1i} & θ_{2i} will both result in the same range, however, $\theta_{2i} = 60°$ with go higher and be in the air longer than $\theta_{1i} = 30°$. Which can be seen in the figure below. (By "air" or course, I mean "the vacuum you can breathe".)
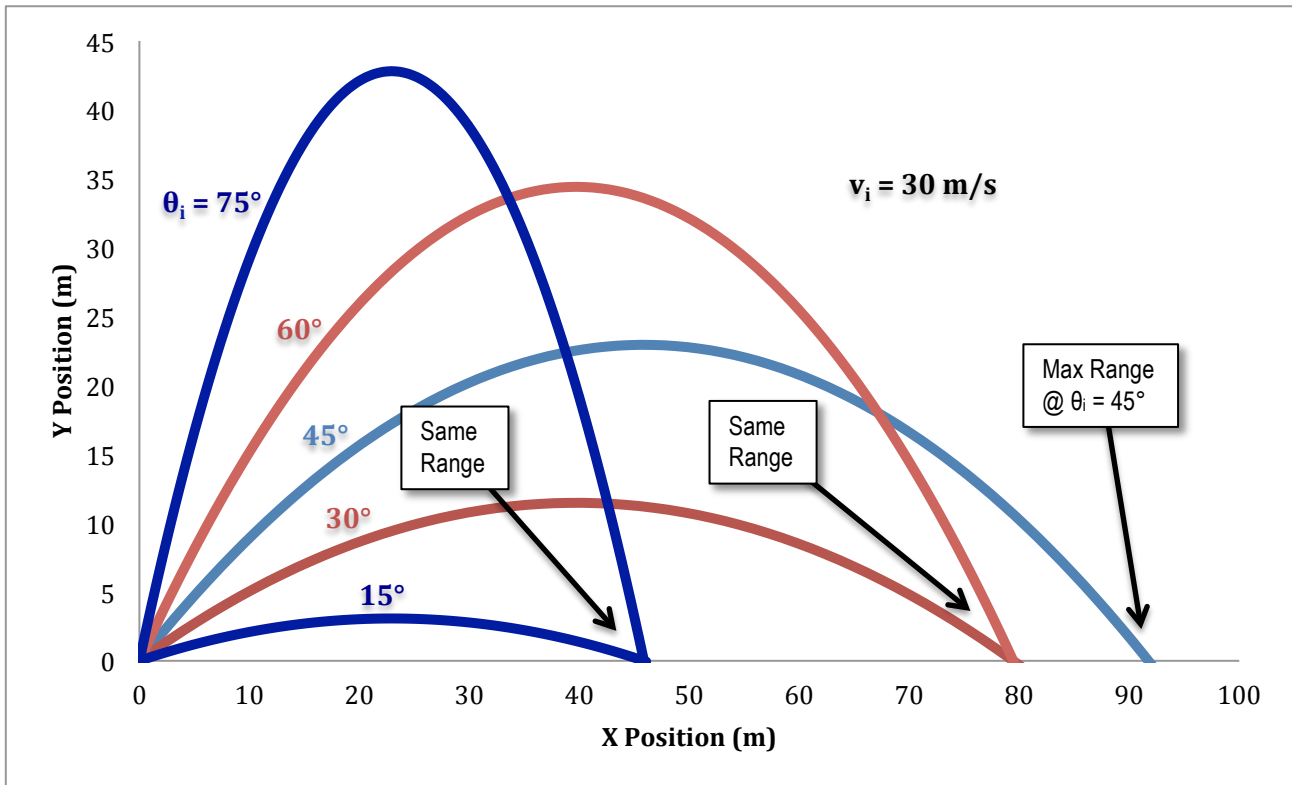
The range of an object in projectile motion means something very specific. It is the displacement in the x direction of an object whose displacement in the y direction is zero. $\Delta x = Range = R$
(in other words, "R", stands for Range.)

The Range Equation or $R = \dfrac{v_i^2 \sin(2\theta_i)}{g}$ can be

derived from the projectile motion equations. We start by breaking our initial velocity in to its components and then list everything we know in the x and y directions:

$$\sin\theta = \frac{O}{H} \Rightarrow \sin\theta_i = \frac{v_{iy}}{v_i} \Rightarrow v_{iy} = v_i \sin\theta_i \,\&$$

$$\cos\theta = \frac{A}{H} \Rightarrow \cos\theta_i = \frac{v_{ix}}{v_i} \Rightarrow v_{ix} = v_i \cos\theta_i = v_x$$

Remember that in the x-direction an object in projectile motion has a constant velocity, therefore $v_{ix} = v_x$.

x-direction: $v_{ix} = v_i \cos\theta_i = v_x$, $\Delta x = R = ?$

y-direction: $\Delta y = 0$ & $a_y = -g$ (remember $g_{Earth} = +9.81\dfrac{m}{s^2}$)

Let's start in the x-direction where there is a constant velocity and solve for the Range.

$$v_x = \frac{\Delta x}{\Delta t} \Rightarrow \Delta x = R = (\Delta t)v_x = (\Delta t)v_i \cos\theta_i$$

Now we need to solve for Δt in the y-direction and substitute Δt in to $R = (\Delta t)v_i \cos\theta$

$$\Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 = 0 \Rightarrow 0 = v_{iy} + \frac{1}{2}a_y\Delta t \Rightarrow v_{iy} = -\frac{1}{2}a_y\Delta t = -\frac{1}{2}(-g)\Delta t = \frac{1}{2}g\Delta t$$
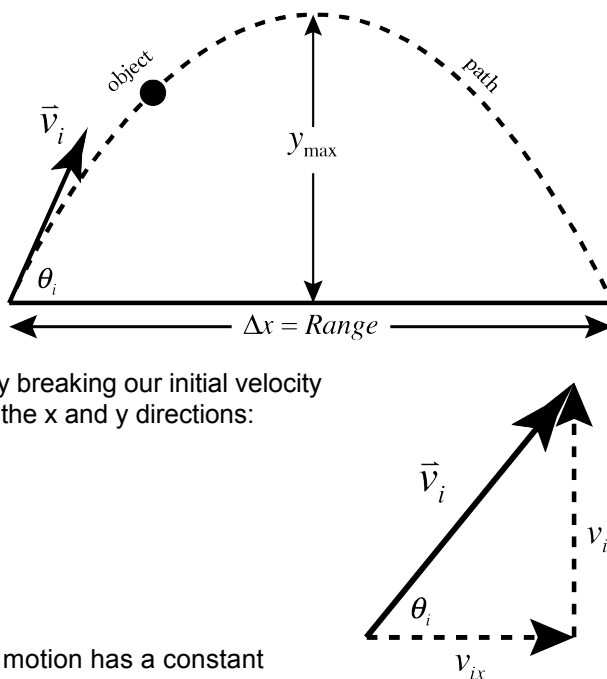
$$\Rightarrow 2v_{iy} = g\Delta t \Rightarrow \Delta t = \frac{2v_{iy}}{g} = \frac{2v_i \sin\theta_i}{g}$$

And now we can substitute back in.

$$R = (\Delta t)v_i \cos\theta_i = \left(\frac{2v_i \sin\theta_i}{g}\right)v_i \cos\theta_i = \frac{v_i^2(2\sin\theta_i \cos\theta_i)}{g} \Rightarrow \boxed{R = \frac{v_i^2 \sin(2\theta_i)}{g}}$$

This uses the sine double angle formula from trig: $2\sin\theta_i \cos\theta_i = \sin(2\theta_i)$

FYI: It is generally *not* assumed that students in an algebra based physics class will know or remember various trig functions like this.

Flipping Physics Lecture Notes:
A Range Equation Problem with Two Parts

Example Problem: Mr.p throws a ball toward a bucket that is 581 cm away from him horizontally. He throws the ball at an initial angle of 55° above the horizontal and the ball is 34 cm short of the bucket. If mr.p throws the ball with the same initial speed and the ball is always released at the same height as the top of the bucket, at what angle does he need to throw the ball so it will land in the bucket?
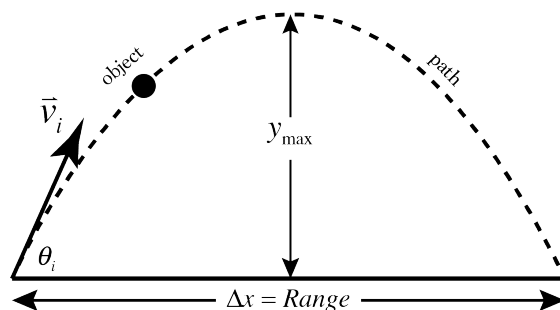
Attempt #1: The range of the first throw is 34 cm short of the bucket.

So $R_1 = 581 - 34 = 547cm \times \dfrac{1m}{100cm} = 5.47m$; $\theta_{1i} = 55°$; $g = +9.81\dfrac{m}{s^2}$; $v_{1i} = ? = v_{2i} = v_i$ & $\theta_{2i} = ?$

Now we need to solve the Range equation for the magnitude of the initial velocity:

$$R_1 = \dfrac{v_i^2 \sin(2\theta_{1i})}{g} \Rightarrow gR_1 = v_i^2 \sin(2\theta_{1i}) \Rightarrow v_i^2 = \dfrac{gR_1}{\sin(2\theta_{1i})} \Rightarrow v_i = \sqrt{\dfrac{gR_1}{\sin(2\theta_{1i})}}$$

$$\Rightarrow v_i = \sqrt{\dfrac{(9.81)(5.47)}{\sin(2 \cdot 55)}} = 7.55675 \dfrac{m}{s}$$

For the 2nd attempt, we use the same initial velocity magnitude, we know the range for the 2nd attempt,

$R_2 = 581cm \times \dfrac{1m}{100cm} = 5.81m$, and now we solve the Range Equation for the angle:

$$R_2 = \dfrac{v_i^2 \sin(2\theta_{2i})}{g} \Rightarrow gR_2 = v_i^2 \sin(2\theta_{2i}) \Rightarrow \sin(2\theta_{2i}) = \dfrac{gR_2}{v_i^2} \Rightarrow \sin^{-1}(\sin(2\theta_{2i})) = 2\theta_{2i} = \sin^{-1}\left(\dfrac{gR_2}{v_i^2}\right)$$

$$\Rightarrow 2\theta_{2i} = \sin^{-1}\left(\dfrac{(9.81)(5.81)}{(7.55675)^2}\right) = 86.46870° \Rightarrow \theta_{2i} = \dfrac{86.46870°}{2} = 43.23435° \approx \boxed{43°}$$

Please note: As shown in "Understanding the Range Equation of Projectile Motion", there are actually two angles that will result in the same range and they are complimentary angles, therefore:

$$\theta_{2i} = 43.23435° \ or \ (90° - 43.23435°) = 43.23435° \ or \ 46.76565° \approx \boxed{43° \ or \ 47°}$$

However, many physics teachers will not require that you understand that there are two angles. Why not? Because knowing that there are two angles with the same range has more to do with your understanding trigonometry than physics.

Example Problem: One bullet is fired horizontally and simultaneously a second bullet is dropped from the same height. Neglecting air resistance and assuming the ground is level, which bullet hits the ground first?

Let's list the known Uniformly Accelerated Motion, UAM, *y-direction* variables for both bullets:

Dropped:	*Fired Horizontally:*	
$\Delta y = -h$	$\Delta y = -h$	The Same
$v_{iy} = 0$	$v_{iy} = 0$	The Same
$a_y = -g$	$a_y = -g$	The Same

If three of the UAM variables are the same, then so are the other two: Δt and v_{fy} are the same for both bullets. Mathematically, it looks like this:

$$\Delta y = v_{iy}\Delta t + \frac{1}{2}a_y\Delta t^2 \Rightarrow -h = (0)\Delta t + \frac{1}{2}(-g)\Delta t^2 \Rightarrow \Delta t^2 = \frac{2h}{g} \Rightarrow \Delta t = \sqrt{\frac{2h}{g}} \text{ (for both)}$$

$$v_{fy} = v_{iy} + a_y\Delta t = 0 + (-g)\left(\sqrt{\frac{2h}{g}}\right) = -\sqrt{\frac{2g^2h}{g}} = \sqrt{2gh} \text{ (for both)}$$

Note: Even though both bullets will strike the ground at the same time and have the same final velocity in the y-direction, the bullet that was fired horizontally will be moving *faster*. This is because it will also have a velocity in the x-direction, and the final velocity will be the resultant vector of both the x and y-direction final velocities. Because the hypotenuse of a right triangle is, by definition, longer than either of the two sides, the one fired horizontally will have a larger magnitude final velocity and, therefore, will be moving faster than the one that was dropped.
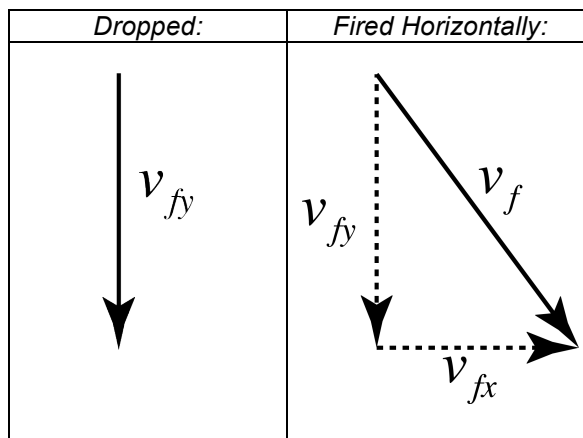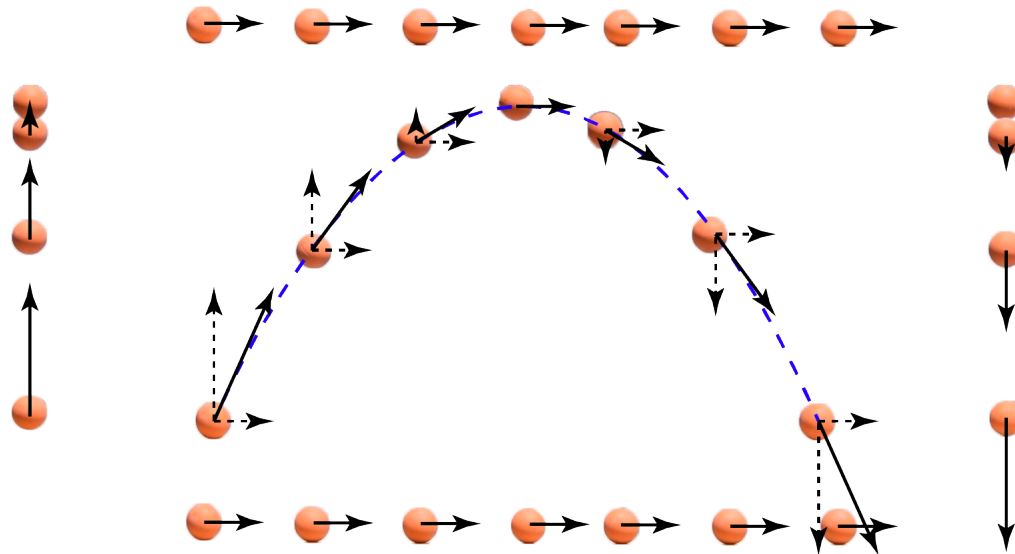
Projectile Motion:
-     X-Direction: Constant Velocity
-     Y-Direction: Uniformly Accelerated Motion (UAM) with a_y = -9.81 m/s^2

The velocity vectors in the x-direction will have a constant value.
The velocity vectors in the y-direction on the way up (on the left) will have a decreasing magnitude upward and on the way down (on the right) will have an increasing magnitude downward. The velocity at the top is zero.
Those vectors are the components of the resultant velocity vectors for projectile motion.

The acceleration of the projectile (in yellow) is always straight down and has a magnitude of 9.81 m/s^2.