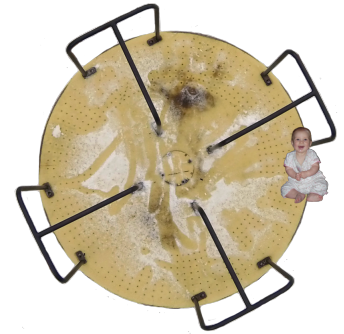Flipping Physics Lecture Notes:

Merry-Go-Round
Conservation of Angular Momentum Problem

Example: A 25 kg child is sitting on the edge of a merry-go-round. The merry-go-round has a mass of 255 kg and is rotating at 2.0 radians per second. The child crawls to the middle of the merry-go-round. What is the final angular speed of the merry-go-round? You may make the following estimations: The child is a point particle; the merry-go-round is a solid disk and has an axle with negligible friction. I_{disk} = ½MR².

Knowns: $m_c = 25 kg$; $m_w = 255 kg$; $\omega_{wi} = 2.0 \dfrac{rad}{s}$; child: outside edge → center; $I_{disk} = \dfrac{1}{2} MR^2$;

child = point particle; $\mu_w \approx 0$; $\omega_{wf} = ?$

For subscripts let's use c for child and w for wheel (a.k.a merry-go-round). Because net torque equals change in angular momentum over change in time, we know angular momentum is conserved when the net external torque acting on the system equals zero. There is no friction on the axle of the merry-go-round, so, because it equals zero, friction on the axle does not cause an external torque on the merry-go-round. As the child crawls toward its center, there is a force of static friction from the merry-go-round on the child and, according to Newton's third law, an equal but opposite force of static friction from the child on the merry-go-round. Because the axis of rotation is at the center of the merry-go-round, those two forces cause equal but opposite torques on the child and merry-go-round system, which means the two torques are internal to the system and cancel one another out. So the net torque on the system is zero and angular momentum, about the axle of the merry-go-round, is conserved.

$$\sum \vec{\tau}_{external} = \frac{\Delta \vec{L}}{\Delta t} = 0 \Rightarrow \sum \vec{L}_i = \sum \vec{L}_f$$

- There is no direction given in the problem and we are solving for angular speed. So let's drop the vector symbol from the equations and solve for the magnitude of final angular velocity.
- The equation for angular momentum of a rigid object with shape is: $\vec{L} = I\vec{\omega}$.
- When a point particle is moving in circular motion, the point particle has an angular momentum: $\vec{L} = I\vec{\omega}$. We will assume this to be true for now, however, we will prove this in a later lesson.

$$\sum \vec{L}_i = \sum \vec{L}_f \Rightarrow L_{ci} + L_{wi} = L_{cf} + L_{wf} \Rightarrow I_{ci}\omega_{ci} + I_w\omega_{wi} = I_{cf}\omega_{cf} + I_w\omega_{wf}$$

- Notice that we do not need subscripts for initial and final for the rotational inertia of the wheel/merry-go-round because its rotational inertia does not change.

$$I_{\substack{system\ of \\ particles}} = \sum_i m_i r_i^2 \Rightarrow I_c = m_c r_c^2$$

- Both objects have the same angular velocities: $\omega_{ci} = \omega_{wi} = \omega_i$ & $\omega_{cf} = \omega_{wf} = \omega_f$
- For the radius of the wheel, let's use R: $r_w = R$

$$\Rightarrow \left(m_c r_{ci}^2 \right)\omega_i + \left(\frac{1}{2} m_w R^2 \right)\omega_i = \left(m_c r_{cf}^2 \right)\omega_f + \left(\frac{1}{2} m_w R^2 \right)\omega_f$$

- The initial distance from the axis of rotation to the location of the child is the same as the radius of the wheel: $r_{ci} = R$.

- The final distance from the axis of rotation to the location of the child is zero: $r_{cf} = 0$.
  - Realize this means our child has zero rotational inertia when sitting at the center of the merry-go-round. Hopefully you realize this will not be true in real life because the child has non-zero size and therefore will have rotational inertia. This is a simplified solution and helps with understanding. The child's small size and therefore quite small rotational inertia relative to the merry-go-round makes this an okay estimation.

$$\Rightarrow m_c \cancel{R^2} \omega_i + \left(\frac{1}{2} m_w \cancel{R^2} \right) \omega_i = m_c \cancel{(0)^2} \omega_f + \left(\frac{1}{2} m_w \cancel{R^2} \right) \omega_f$$

- Everybody brought R^2 to the party!

$$\Rightarrow m_c \omega_i + \frac{1}{2} m_w \omega_i = \frac{1}{2} m_w \omega_f \Rightarrow 2 m_c \omega_i + m_w \omega_i = m_w \omega_f \Rightarrow \omega_f = \frac{\omega_i (2 m_c + m_w)}{m_w}$$

$$\omega_f = \frac{(2)\left[(2)(25) + 255 \right]}{255} = 2.39216 \approx \boxed{2.4 \frac{rad}{s}}$$

Does it make sense that the angular velocity of the system increases as the child moves toward the middle of the merry-go-round? Let's go back to the conservation of angular momentum equation near the beginning of the solution:

$$\Rightarrow I_{ci} \omega_{ci} + I_w \omega_{wi} = I_{cf} \omega_{cf} + I_w \omega_{wf} \Rightarrow I_{ci} \omega_i + I_w \omega_i = I_{cf} \omega_f + I_w \omega_f \Rightarrow \left(I_{ci} + I_w \right) \omega_i = \left(I_{cf} + I_w \right) \omega_f$$

Angular momentum is conserved. The rotational inertia of the wheel remains unchanged throughout the whole event. But as the child moves in toward the axis of rotation, the rotational inertia of the child decreases, therefore, in order to maintain a constant angular momentum of the system, the angular velocity of the system has to increase. In other words, the angular momentum of the system does not change, so if the rotational inertia of the system decreases, the angular velocity of the system must increase.

Realize, this increase in angular velocity of the system actually represents an increase in the kinetic energy of the system. There is no translational motion of the system, so all of the kinetic energy is rotational kinetic energy. The change in kinetic energy of the system is:

$$\Delta KE = KE_f - KE_i = KE_{cf} + KE_{wf} - KE_{ci} - KE_{wi}$$

$$\Rightarrow \Delta KE = \frac{1}{2} I_{cf} \omega_f^2 + \frac{1}{2} I_w \omega_f^2 - \frac{1}{2} I_{ci} \omega_i^2 - \frac{1}{2} I_w \omega_i^2 = \frac{1}{2} (0) \omega_f^2 + \frac{1}{2} \left(\frac{1}{2} m_w R^2 \right) \omega_f^2 - \frac{1}{2} \left(m_c R^2 + \frac{1}{2} m_w R^2 \right) \omega_i^2$$

$$\Rightarrow \Delta KE = R^2 \left[\frac{1}{4} m_w \omega_f^2 - \left(\frac{1}{2} m_c + \frac{1}{4} m_w \right) \omega_i^2 \right]$$

$$\Rightarrow \Delta KE = R^2 \left[\frac{1}{4} (255)(2.39216)^2 - \left(\frac{1}{2} 25 + \frac{1}{4} 255 \right) (2)^2 \right] = 59.804 R^2 \approx 6.0 \times 10^1 R^2 = W_{net}$$

The change in kinetic energy of the system is positive because the child had to do work on herself, and therefore the system, in order to crawl from the outside edge to the center of the merry-go-round. Because net work equals change in kinetic energy, we know she did $60 R^2$ joules of work on the system to increase the kinetic energy of the system. The larger the radius of the merry-go-round, the more work she has to do to the system.