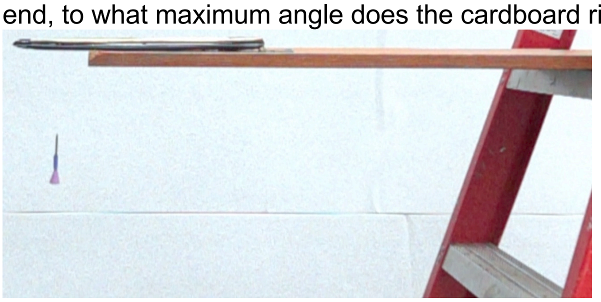
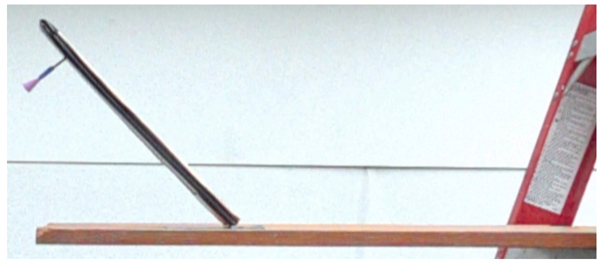
Flipping Physics Lecture Notes:

Dart with Thin Rod Collision
Conservation of Angular Momentum Demonstration and Problem

Example: As shown, a 5.3 g dart is moving vertically at 16.5 m/s just before it collides with and sticks to a 33.9 cm long, thin piece of cardboard. If the dart hits the 71.8 g piece of cardboard 28.7 cm from its fixed end, to what maximum angle does the cardboard rise? $I_{end} = \frac{1}{3}mL^2$





Knowns: $m_d = 0.0053kg;\ v_{di} = 16.5\frac{m}{s};\ m_c = 0.0718kg;\ x = 0.287m; L_c = 0.339m;\ \theta_f = ?$

$$I_{end} = \frac{1}{3}mL^2 = I_c = \frac{1}{3}(0.0718)(0.339)^2 = 0.0027504kg \cdot m^2$$

We have already determined that, about the axis of rotation of the rigid object with shape, angular momentum is conserved in a situation where a point particle collides with a rigid object with shape. https://www.flippingphysics.com/collision-angular-momentum-conservation.html

However, the collision is just part 1 of the problem, part 2 of the problem is conservation of mechanical energy as the cardboard and dart rise together. We know mechanical energy is conserved while the cardboard rotates because there is no energy added to or removed from the system via work done by a force of friction or force applied.

$$\sum \vec{\tau}_{\substack{\text{dart and cardboard} \\ \text{AoR @ fixed end of cardboard}}} = \frac{\Delta \vec{L}}{\Delta t} = 0 \Rightarrow \sum \vec{L}_{1i} = \sum \vec{L}_{1f} \Rightarrow L_{d1i} + L_{c1i} = L_{d1f} + L_{c1f} \ \&\ ME_{2i} = ME_{2f}$$

- We currently have two equations for angular momenta of objects: $L_{\text{point partilce}} = rmv\sin\theta \ \&\ L_{\substack{\text{Rigid Object} \\ \text{with Shape}}} = I\omega$
- For the dart before the collision, we use the equation for the angular momentum of a point particle: $L_{d1i} = r_{d1i}m_d v_{d1i}\sin\theta_{d1i}$
    - We are treating the dart as a point particle, so you will see me refer to it as both a "dart" and a "point particle", depending on the situation.
- Before the collision the cardboard is not moving, so it does not have any angular momentum.
- For the dart after the collision, we have shown that we can use either of those two equations for the angular momentum of the point particle because it is moving in a circle. Let's use: $L_{d1f} = I_{d1f}\omega_{d1f}$
  https://www.flippingphysics.com/point-particle-kinetic-energy-and-angular-momentum-equations.html

- For the cardboard after the collision, we have to use the equation for the angular momentum of a rigid object with shape: $L_{c1f} = I_c \omega_{c1f}$

$$\Rightarrow r_{d1i} m_d v_{d1i} \sin\theta_{d1i} = I_{d1f}\omega_{d1f} + I_c \omega_{c1f}$$

- We have already shown that, while r_{id} and θ_{id} change as the dart moves, x, which equals $r_{d1i}\sin\theta_{d1i}$ does not change: $\sin\theta_{d1i} = \frac{O}{H} = \frac{x}{r_{d1i}} \Rightarrow x = r_{d1i}\sin\theta_{d1i}$
https://www.flippingphysics.com/angular-momentum-triangle.html
- The equation for rotational inertia of a point particle is: $I_{d1f} = m_d r_{d1f}^{\ 2}$

$$\Rightarrow x m_d v_{d1i} = m_d r_{d1f}^{\ 2}\omega_{d1f} + I_c \omega_{c1f}$$

- The dart stays the distance "x" from the axis of rotation after it sticks to the cardboard: $r_{d1f} = x$
- Both the dart and the cardboard are rotating together after the collision and the final angular velocity of part 1 is the same as the initial angular velocity for part 2: $\omega_{d1f} = \omega_{c1f} = \omega_{1f} = \omega_{2i}$

$$\Rightarrow x m_d v_{d1i} = m_d x^2 \omega_{1f} + I_c \omega_{1f} = \left(m_d x^2 + I_c\right)\omega_{1f} \Rightarrow \omega_{1f} = \frac{x m_d v_{d1i}}{m_d x^2 + I_c} = \omega_{2i}$$

$$\Rightarrow \omega_{1f} = \frac{(0.287)(0.0053)(16.5)}{(0.0053)(0.287)^2 + 0.0027504} = 7.87517\frac{rad}{s} = \omega_{2i}$$

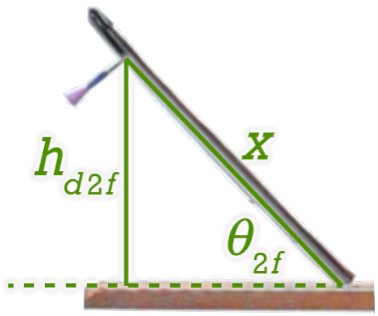
- And now for part 2 we use conservation of mechanical energy: $ME_{2i} = ME_{2f}$
  - The initial point is right after the dart collides with the cardboard and before the cardboard starts to rise. The final point is at the maximum angle the cardboard rises to. Lets set the horiztonal zero line at the initial height of the dart and cardboard.
- There are no springs, so no elastic potential energy in the problem.
- Initially the dart and cardboard are at the horizontal zero line, so they do not have any gravitational potential energy initial.
- The only type of initial mechanical energy is kinetic energy and because the system is rotating, it is the combined rotational kinetic energy initial of both the dart and the cardboard.
- At the final point, the dart and cardboard are not moving, so they have no final kinetic energy.
- At the final point, the dart and cardboard are above the zero line, so they have final gravitational potential energy.

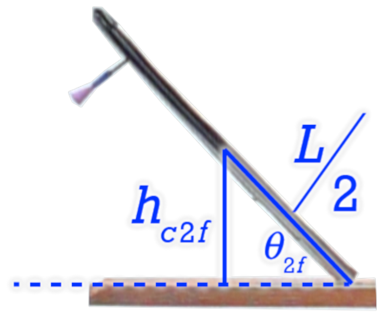
$$\Rightarrow \frac{1}{2} I_{system}\omega_{2i}^{\ 2} = m_d g h_{d2f} + m_c g h_{c2f}$$

- The rotational inertia of the system is: $I_{system} = I_d + I_c = m_d r_{d2f}^{\ 2} + I_c = m_d x^2 + I_c$
  - The distance from the dart to the axis of rotation is still x: $r_{d2f} = x$

$$\Rightarrow \frac{1}{2}\left(m_d x^2 + I_c\right)\omega_{2i}^{\ 2} = m_d g h_{d2f} + m_c g h_{c2f}$$

- Notice the only two variables we do not know in this equation are the two height finals.
    - In order to reduce this equation to one unknown, we need to draw triangles and use trigonometry to find relationships between the angle, θ, between the cardboard and the horizontal, and both height finals.
    - The height for part 2 final of the cardboard is to the center mass of the cardboard. We will consider the cardboard to be of uniform density, so the center of mass is in the middle of the cardboard.





$$\sin\theta_{2f} = \frac{O}{H} = \frac{h_{d2f}}{x} \Rightarrow h_{d2f} = x\sin\theta_{2f} \;\&\; \sin\theta_{2f} = \frac{O}{H} = \frac{h_{c2f}}{L/2} \Rightarrow h_{c2f} = \frac{L}{2}\sin\theta_{2f}$$

$$\Rightarrow \frac{1}{2}\left(m_d x^2 + I_c\right)\omega_{2i}^{\,2} = m_d g x \sin\theta_{2f} + m_c g \frac{L}{2}\sin\theta_{2f} = \left(m_d g x + m_c g \frac{L}{2}\right)\sin\theta_{2f}$$

$$\Rightarrow \sin\theta_{2f} = \frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\omega_{2i}^{\,2}}{m_d g x + m_c g \frac{L}{2}} \Rightarrow \theta_{2f} = \sin^{-1}\left(\frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\omega_{2i}^{\,2}}{m_d g x + m_c g \frac{L}{2}}\right)$$

$$\Rightarrow \theta_{2f} = \sin^{-1}\left(\frac{\left(\frac{1}{2}\right)\left[(0.0053)(0.287)^2 + 0.0027504\right](7.87517)^2}{(0.0053)(9.81)(0.287) + (0.0718)(9.81)\left(\frac{0.339}{2}\right)}\right) = 48.101° \approx \boxed{48°}$$

This is our predicted final angle of the cardboard.

$$\text{Measured } \theta_{2f} = 46°$$

$$\text{Percent Difference} = \frac{48.101 - 46}{(48.101 + 46)/2} \times 100 = 4.46603 \approx 4.5\%$$

Alternative solution with not solving for the initial angular velocity for part 2:

$$\omega_{2i} = \frac{x m_d v_{d1i}}{m_d x^2 + I_c} \text{ \& } \theta_{2f} = \sin^{-1}\left(\frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\omega_{2i}^{\ 2}}{m_d g x + m_c g \frac{L}{2}}\right) = \sin^{-1}\left(\frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\left(\frac{x m_d v_{d1i}}{m_d x^2 + I_c}\right)^2}{m_d g x + m_c g \frac{L}{2}}\right)$$

$$\Rightarrow \theta_{2f} = \sin^{-1}\left(\frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\frac{\left(x m_d v_{d1i}\right)^2}{\left(m_d x^2 + I_c\right)^2}}{m_d g x + m_c g \frac{L}{2}}\right) = \sin^{-1}\left(\frac{\frac{\left(x m_d v_{d1i}\right)^2}{m_d x^2 + I_c}}{2 m_d g x + m_c g L}\right)$$

$$\Rightarrow \theta_{2f} = \sin^{-1}\left(\frac{\left(x m_d v_{d1i}\right)^2}{\left(m_d x^2 + I_c\right)\left(2 m_d g x + m_c g L\right)}\right)$$