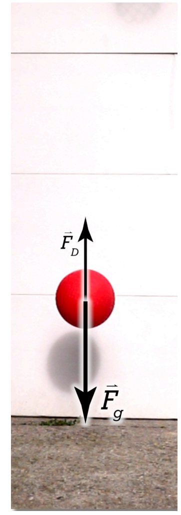
Flipping Physics Lecture Notes:

Deriving Drag Force Motion Equations

http://www.flippingphysics.com/drag-force-motion-equations.html



We are going to use calculus to derive the equations of motion for an object with an initial velocity of zero and a drag force acting on it described using the equation: $\vec{F}_D = -b\vec{v}$

Let's define down, the direction the object is moving, as positive. Believe it or not, this makes the math easier when we get further into the problem.

We can use Newton's Second Law to determine the terminal velocity of the ball.

$$\sum F_y = F_g - F_D = ma_y = m(0) = 0 \Rightarrow F_D = F_g \Rightarrow bv_t = mg \Rightarrow v_t = \frac{mg}{b}$$

Please realize the ball in our demonstration never gets close to its terminal velocity.

We can also use Newton's Second Law to determine the velocity of the ball as a function of time:

$$\sum F_y = F_g - F_D = ma_y \Rightarrow mg - bv = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = g - \frac{bv}{m} \Rightarrow \frac{1}{g - \frac{bv}{m}}dv = dt \Rightarrow \int_0^v \frac{1}{g - \frac{bv}{m}}dv = \int_0^t dt$$

$$\text{let } u = g - \frac{bv}{m} \Rightarrow du = -\frac{b}{m}dv \Rightarrow dv = -\frac{m}{b}du \;\&\; u_i = g - \frac{b(0)}{m} = g \;\&\; u_f = g - \frac{bv}{m} = u \;\&\; \int \frac{1}{x+a}dx = \ln|x+a|$$

$$\Rightarrow \int_g^u \frac{1}{u}\left(-\frac{m}{b}\right)du = -\left(\frac{m}{b}\right)\int_g^u \frac{1}{u}du = \int_0^t dt \Rightarrow -\left(\frac{m}{b}\right)\left[\ln u\right]_g^u = \left[t\right]_0^t \Rightarrow -\left(\frac{m}{b}\right)\left[\ln\left(g - \frac{bv}{m}\right)\right]_0^v = \left[t\right]_0^t$$

$$\Rightarrow -\left(\frac{m}{b}\right)\left[\ln\left(g - \frac{bv}{m}\right) - \ln\left(g - \frac{b(0)}{m}\right)\right] = -\left(\frac{m}{b}\right)\left[\ln\left(g - \frac{bv}{m}\right) - \ln(g)\right] = -\left(\frac{m}{b}\right)\ln\left(\frac{g - \frac{bv}{m}}{g}\right) = t - 0 = t$$

$$\ln A - \ln B = \ln\left(\frac{A}{B}\right) \;\&\; e^{\ln(x)} = x$$

$$\int_0^v \frac{1}{g - \frac{bv}{m}}dv = \int_0^t dt \Rightarrow \int_0^v \frac{1}{\left(-\frac{b}{m}\right)\left(v - \frac{mg}{b}\right)}dv = \left(-\frac{m}{b}\right)\int_0^v \frac{1}{\left(v - \frac{mg}{b}\right)}dv = \left(-\frac{m}{b}\right)\ln$$

$$\ln\left(\frac{g - \frac{bv}{m}}{g}\right) = -\frac{bt}{m} \Rightarrow e^{\ln\left(\frac{g - \frac{bv}{m}}{g}\right)} = e^{-\frac{bt}{m}} \Rightarrow \frac{g - \frac{bv}{m}}{g} = e^{-\frac{bt}{m}} \Rightarrow g - \frac{bv}{m} = ge^{-\frac{bt}{m}} \Rightarrow \frac{bv}{m} = g - ge^{-\frac{bt}{m}}$$

$$\Rightarrow v = \frac{mg}{b} - \frac{mg}{b}e^{-\frac{bt}{m}} = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right) \Rightarrow v_t\left(1 - e^{-\frac{bt}{m}}\right)$$

Actually, you do not have to use u substitution if you do not want to:

$$\int_0^v \frac{1}{g - \frac{bv}{m}}dv = \int_0^v \frac{1}{\left(-\frac{b}{m}\right)\left(v - \frac{mg}{b}\right)}dv = \left(-\frac{m}{b}\right)\int_0^v \frac{1}{v - \frac{mg}{b}}dv = -\left(\frac{m}{b}\right)\left[\ln\left(v - \frac{mg}{b}\right)\right]_0^v = -\left(\frac{m}{b}\right)\left[\ln\left(v - \frac{mg}{b}\right) - \ln\left(0 - \frac{mg}{b}\right)\right]$$

$$\Rightarrow -\left(\frac{m}{b}\right)\ln\left(\frac{v-\frac{mg}{b}}{-\frac{mg}{b}}\right) = -\left(\frac{m}{b}\right)\ln\left(\frac{\left(-\frac{b}{m}\right)v+g}{g}\right) = -\left(\frac{m}{b}\right)\ln\left(\frac{g-\frac{bv}{m}}{g}\right) \;\&\; \int\frac{1}{x+a}dx = \ln\left|x+a\right| \text{ where } x = v \;\&\; a = -\frac{mg}{b}$$

Notice this fits our initial condition that the initial velocity equals zero:

$$v(0) = v_t\left(1-e^{-\frac{b(0)}{m}}\right) = v_t\left(1-e^{-0}\right) = v_t(1-1) = 0$$

It also gives us the same equation for terminal velocity:

$$v(\infty) = v_t\left(1-e^{-\frac{b(\infty)}{m}}\right) = v_t\left(1-e^{-\infty}\right) = v_t(1-0) = v_t$$

And we can take the derivative with respect to time of the velocity equation to get acceleration as a function of time:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left[\frac{mg}{b}\left(1-e^{-\frac{bt}{m}}\right)\right] = \frac{d}{dt}\left[\frac{mg}{b}\right] - \frac{d}{dt}\left[\frac{mg}{b}e^{-\frac{bt}{m}}\right] = 0-\left(\frac{mg}{b}\right)\left(-\frac{b}{m}\right)e^{-\frac{bt}{m}} \Rightarrow a(t) = ge^{-\frac{bt}{m}}$$

$$\frac{d}{dx}e^{ax} = ae^{ax} \Rightarrow \frac{d}{dt}e^{-\frac{bt}{m}} = \left(-\frac{b}{m}\right)e^{-\frac{bt}{m}}$$ (using the chain rule) The Chain Rule $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

This fits with our initial condition that the ball starts in free fall with an acceleration of magnitude g:

$$a(0) = ge^{-\frac{b(0)}{m}} = g$$

And with the final condition that the ball will reach terminal velocity and have zero acceleration:

$$a(\infty) = ge^{-\frac{b(\infty)}{m}} = 0$$

And, we can take the integral with respect to time of our velocity equation to get the position of the ball as a function of time. Setting our initial position as zero, our initial time as zero, our final position as y and our final time as t:

$$v = \frac{dy}{dt} = v_t\left(1-e^{-\frac{bt}{m}}\right) \Rightarrow dy = \left[v_t\left(1-e^{-\frac{bt}{m}}\right)\right]dt \Rightarrow \int_0^y dy = v_t\int_0^t\left(1-e^{-\frac{bt}{m}}\right)dt$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} \Rightarrow \int e^{-\frac{bt}{m}}\,dt = \left(-\frac{m}{b}\right)e^{-\frac{bt}{m}}$$

$$\left[y\right]_0^y = v_t\left[t-\left(-\frac{m}{b}\right)e^{-\frac{bt}{m}}\right]_0^t = v_t\left[t+\left(\frac{m}{b}\right)e^{-\frac{bt}{m}}\right]_0^t \Rightarrow y-0 = v_t\left[\left(t+\left(\frac{m}{b}\right)e^{-\frac{bt}{m}}\right)-\left(0+\left(\frac{m}{b}\right)e^{-\frac{b(0)}{m}}\right)\right]$$

$$\Rightarrow y = v_t\left[t+\left(\frac{m}{b}\right)e^{-\frac{bt}{m}}-\frac{m}{b}\right] \Rightarrow y(t) = v_t t+\left(\frac{mv_t}{b}\right)\left[e^{-\frac{bt}{m}}-1\right] = \left(\frac{mg}{b}\right)t+\left(\frac{m^2g}{b^2}\right)\left[e^{-\frac{bt}{m}}-1\right]$$

This fits our initial condition that the initial position equals zero:

$$y(0)=(v_t)(0)+\left(\frac{mv_t}{b}\right)\left[e^{-\frac{b(0)}{m}}-1\right]=\left(\frac{mv_t}{b}\right)[1-1]=0$$

Graphing these equations to 99% of the terminal velocity looks like this: