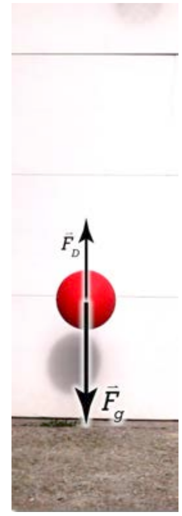
Flipping Physics Lecture Notes:

Deriving Drag Force Motion Equations

http://www.flippingphysics.com/drag-force-time-constant-equations.html



For a dropped object experiencing a drag force, in our previous lesson "Deriving Drag Force Motion Equations"[1], we derived the position, velocity, and acceleration as a function of time, and the object's terminal velocity. It is important to realize we defined down as positive.

$$v_i = 0;\ \vec{F}_D = -b\vec{v};\ y(t) = \left(\frac{mg}{b}\right)t + \left(\frac{m^2 g}{b^2}\right)\left[e^{-\frac{bt}{m}} - 1\right];\ v(t) = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right);\ a(t) = ge^{-\frac{bt}{m}};\ v_t = \frac{mg}{b}$$

Down is positive! τ means *time constant.* (and *torque*, sorry!)

What we are doing today is defining the time constant, τ. τ is the lowercase Greek letter "tau". And yes, it is the same as the symbol for torque. I am sorry about that. Specifically looking at our velocity equation:

$$v(t) = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right) = \frac{mg}{b}\left(1 - e^{-\frac{t}{m/b}}\right) = \frac{mg}{b}\left(1 - e^{-\frac{t}{\tau}}\right)$$

The time constant in this equation equals the mass over the proportionality constant: $\tau = \frac{m}{b}$

The way you determine the time constant is by figuring out what value to put in the denominator of the exponent such that you will get negative t over time constant, τ, in the exponent. There are other time constants for other relationships, so realize the time constant only equals mass over proportionality constant in this specific example.

Let's determine the units[2] for the time constant:
That's right, we get seconds. Did I mention τ is called the "time constant"?

$$\tau = \frac{m}{b} \Rightarrow \tau(\text{units}) = \frac{kg}{kg/s} = s$$

Also notice the terminal velocity is in our velocity equation: $v_t = \frac{mg}{b} \rightarrow v(t) = v_t\left(1 - e^{-\frac{t}{\tau}}\right)$

But we have not actually answered the question "What is the time constant?"
Here we go. Look at what happens when we solve for the object's velocity at one time constant or t = τ.

$$v(\tau) = v_t\left(1 - e^{-\frac{\tau}{\tau}}\right) = v_t\left(1 - e^{-1}\right) = v_t(1 - 0.367879) = 0.632121v_t \Rightarrow v(\tau) \approx 0.632v_t$$

In other words, the time constant is the time it takes for the velocity of the falling object to reach 63.2 percent of its terminal velocity. The time constant is a general way for us to define the time it takes for an exponential equation to get to a percent of its maximum value. The time constant represents the time it takes for a 63.2% change. We can also look at multiples of the time constant.

$$v(2\tau) = v_t\left(1 - e^{-\frac{2\tau}{\tau}}\right) = v_t\left(1 - e^{-2}\right) = v_t(1 - 0.135335) = 0.864665v_t \Rightarrow v(2\tau) \approx 0.865v_t$$

After 2 time constants, the velocity has reached 86.5% of its terminal velocity.
This represents, from 1 to 2 times constants, another increase of 63.2% closer to terminal velocity.
In other words:
(1 - 0.632121) = 0.367879 → 36.7879% or the amount remaining after 1 time constant.
0.367879 x 0.632121 = 0.232544 or 63.2% of the remaining amount
0.632121 + 0.232544 = 0.864665 or 86.5%

[1] http://www.flippingphysics.com/drag-force-motion-equations.html
[2] We derived the units for the proportionality constant in a previous lesson: http://www.flippingphysics.com/drag-force-free-fall.html

$$v(3\tau) = v_t\left(1 - e^{-\frac{3\tau}{\tau}}\right) = v_t\left(1 - e^{-3}\right) = v_t\left(1 - 0.049787\right) = 0.950213 v_t \Rightarrow v(3\tau) \approx 0.950 v_t$$

After 3 time constants, the velocity has reached 95.0% of its terminal velocity. Again, an increase which gets the velocity 63.2% closer to the maximum velocity, each time change of an integer multiple of the time constant gets the value 63.2% closer to the maximum value.

And it takes roughly 6.91 time constants to reach 99.9% of its terminal velocity.

$$v(x\tau) = 0.999 v_t = v_t\left(1 - e^{-\frac{x\tau}{\tau}}\right) \Rightarrow 0.999 = 1 - e^{-x} \Rightarrow 0.001 = e^{-x} \Rightarrow \ln(0.001) = \ln\left(e^{-x}\right) \Rightarrow -6.90776 = -x \Rightarrow x \approx 6.91$$

Please recognize this number of 63.2% is coming up repeatedly. For the purposes of the AP Physics C exams, it is a number you should recognize and know how to determine:

$$1 - e^{-1} = 1 - 0.367879 = 0.632121 \approx 0.632$$

But what does the time constant mean for acceleration? Let's take a look:

$$a(t) = ge^{-\frac{bt}{m}} = ge^{-\frac{t}{m/b}} \Rightarrow a(t) = ge^{-\frac{t}{\tau}} \Rightarrow a(\tau) = ge^{-\frac{\tau}{\tau}} = ge^{-1} = 0.367879g \Rightarrow a(\tau) \approx 0.368g$$

When we recognize the initial acceleration of the object has a magnitude of g:

$$a(0) = ge^{-\frac{0}{\tau}} = ge^{-0} = g$$

We can see that the acceleration has decreased from g to 0.368 g, or a decrease of 63.2%.

And here is what we get when using the time constant with position:

$$y(t) = \left(\frac{mg}{b}\right)t + \left(\frac{m^2 g}{b^2}\right)\left[e^{-\frac{bt}{m}} - 1\right] = \left(\frac{mg}{b}\right)t + \left(\frac{mg}{b}\right)\left(\frac{m}{b}\right)\left[e^{-\frac{t}{m/b}} - 1\right] \Rightarrow y(t) = v_t t + v_t \tau\left[e^{-\frac{t}{\tau}} - 1\right]$$

$$y(\tau) = v_t \tau + v_t \tau\left[e^{-\frac{\tau}{\tau}} - 1\right] = v_t \tau + v_t \tau\left[e^{-1} - 1\right] = v_t \tau + v_t \tau\left[0.367879 - 1\right] = v_t \tau - 0.632121 v_t \tau$$

$$\Rightarrow y(\tau) = 0.367879 v_t \tau \approx 0.368 v_t \tau$$

When we recognize the initial position of the object is 0:

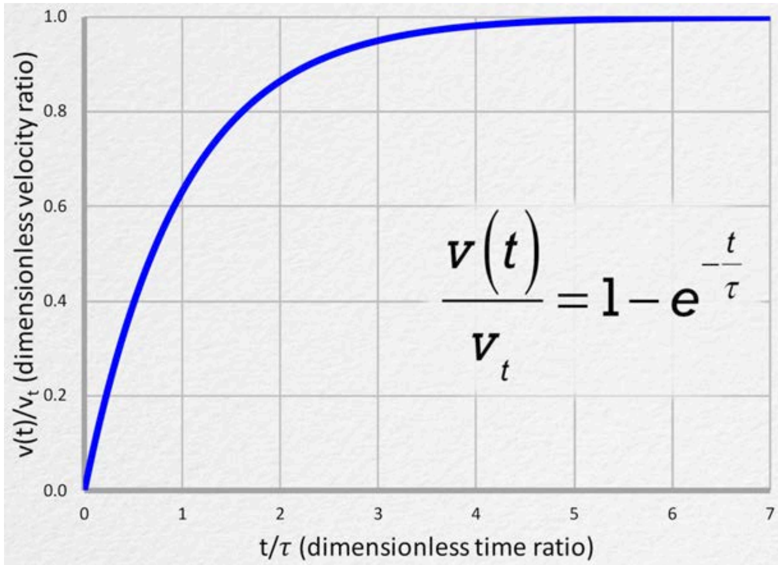
$$y(0) = v_t(0) + v_t(0)\left[e^{-\frac{0}{\tau}} - 1\right] = 0$$

We can see that the position is now 1.632 times the terminal velocity times the time constant. So, while this does not have the same shape as velocity and acceleration, there are some similarities which are easiest to see when we look at the graphs of these functions, starting with velocity.

We are not going to graph velocity as a function of time, instead we are going to rearrange the equation:

$$v(t) = v_t\left(1 - e^{-\frac{t}{\tau}}\right) \Rightarrow \frac{v(t)}{v_t} = 1 - e^{-\frac{t}{\tau}}$$

We have solved for velocity as a function of time divided by terminal velocity. Let's call this a "dimensionless velocity ratio" which we will put on the y-axis. On the x-axis let's put time divided by time constant or a "dimensionless time ratio". The graph looks like this:



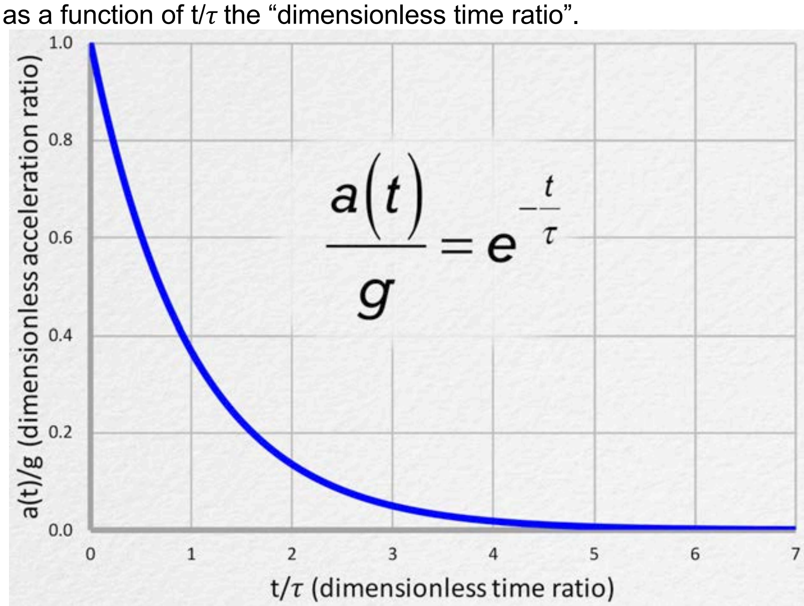
Things to notice about this velocity graph:

- When time equals 1 time constant, we are at the number 1 on the x-axis.
    - At this point the velocity has reached 63.2% of the terminal velocity.
    - That is why the y-axis value at this point is 0.632.
- When time equals 2 times constants, the y-axis value is 0.865, because the velocity has reached 86.5% of its terminal velocity.
- When the time equals 7 time constants, the y-axis value is 0.999, because the velocity has reached 99.9%of its terminal velocity.
- The curve has a horizontal asymptote at 1 because the maximum velocity which the object will reach is the terminal velocity, v_t , and v_t / v_t = 1.

Next let's look at acceleration:

$$a(t)=ge^{-\frac{t}{\tau}} \Rightarrow \frac{a(t)}{g}=e^{-\frac{t}{\tau}}$$

Here we graph acceleration as a function of time over acceleration due to gravity or a "dimensionless acceleration ratio" as a function of t/τ the "dimensionless time ratio".



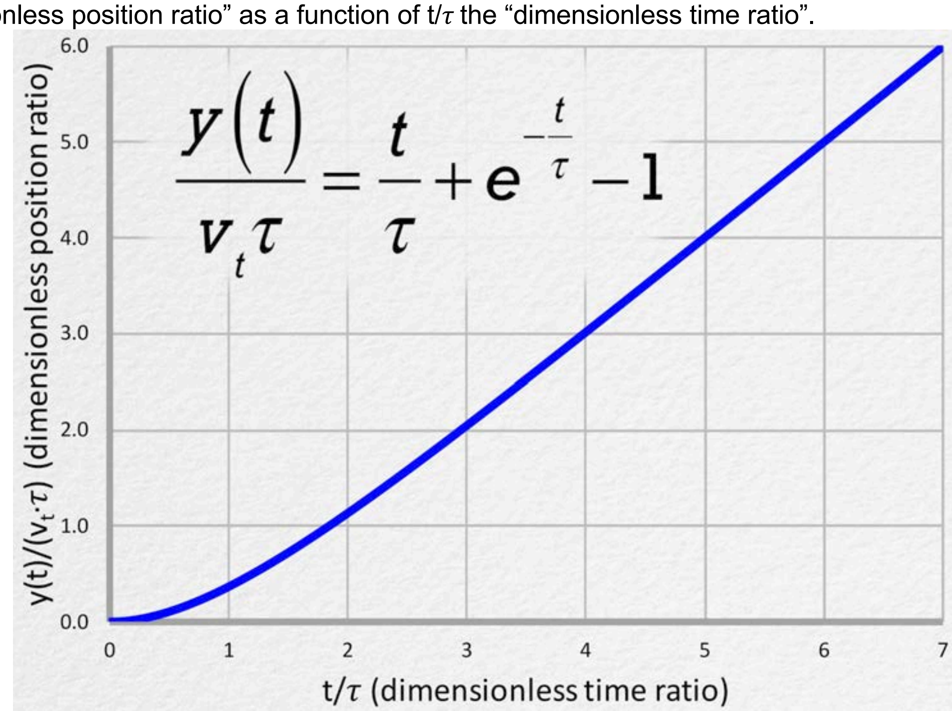
Things to notice about this acceleration graph:

- After 1 time constant the y-axis is at 0.368 because the acceleration has decreased by 63.2%.
- After 7 time constants the y-axis is at 0.001 because the acceleration has decreased by 99.9%
- The curve has a horizontal asymptote of 0 because the minimum acceleration is 0 and 0/g = 0.

Next let's look at position:

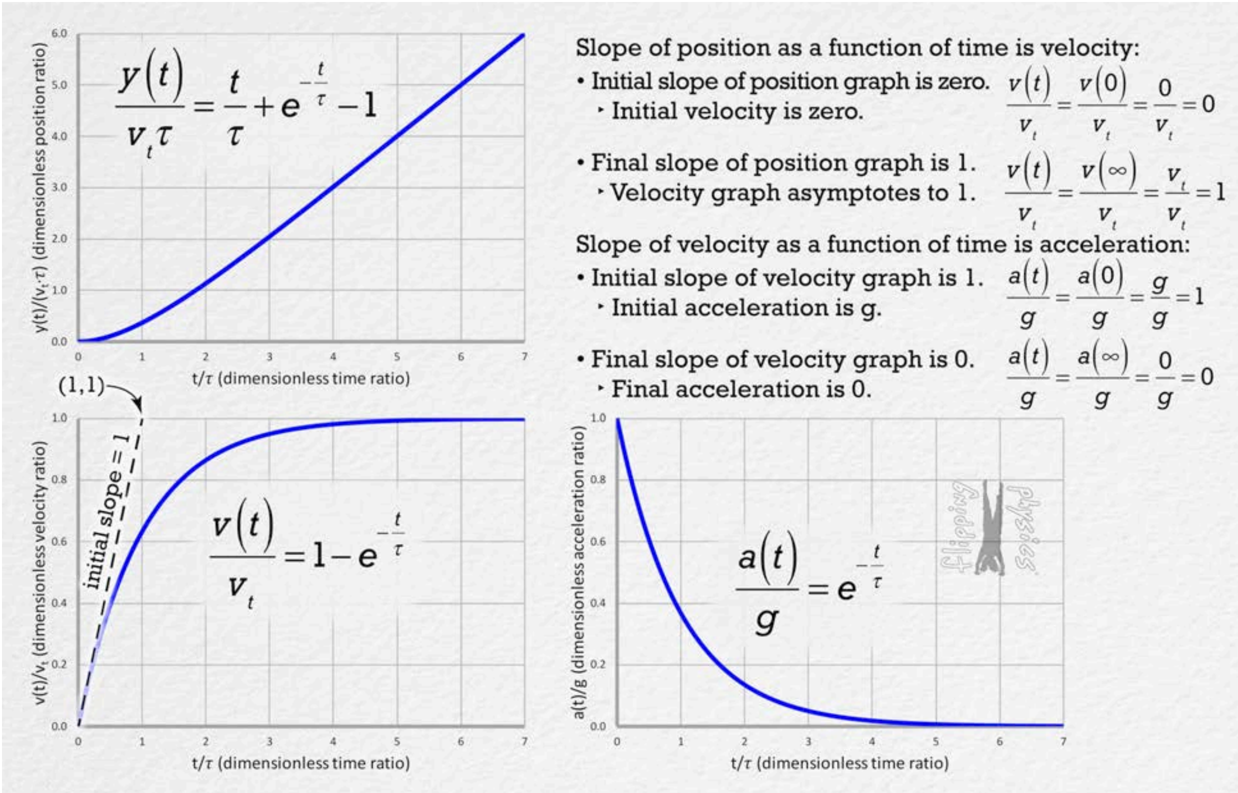
$$y(t) = v_t t + v_t \tau \left[e^{-\frac{t}{\tau}} - 1 \right] \Rightarrow \frac{y(t)}{v_t \tau} = \frac{t}{\tau} + e^{-\frac{t}{\tau}} - 1$$

Here we graph position as a function of time over the quantity terminal velocity times the time constant or a "dimensionless position ratio" as a function of t/τ the "dimensionless time ratio".



Things to notice about this graph:

- The initial y-axis value is zero because the initial position is zero:
  - $$\frac{y(0)}{v_t \tau} = \frac{0}{\tau} + e^{-\frac{0}{\tau}} - 1 = e^{-0} - 1 = 1 - 1 = 0$$
- The curve asymptotes to a slope of 1. In other words, as the velocity approaches its terminal velocity, the distance traveled during each time constant approaches $v_t \cdot \tau$.
  - $$v = \frac{\Delta x}{\Delta t} \Rightarrow v_t = \frac{\Delta x_\tau}{\tau} \Rightarrow \Delta x_\tau = v_t \tau$$


$\frac{y(t)}{v_t\tau}=\frac{t}{\tau}+e^{-\frac{t}{\tau}}-1$
y(t)/(v$_t$·τ) (dimensionless position ratio)
t/τ (dimensionless time ratio)
Slope of position as a function of time is velocity:
• Initial slope of position graph is zero.
‣ Initial velocity is zero.
$\frac{v(t)}{v_t}=\frac{v(0)}{v_t}=\frac{0}{v_t}=0$
• Final slope of position graph is 1.
‣ Velocity graph asymptotes to 1.
$\frac{v(t)}{v_t}=\frac{v(\infty)}{v_t}=\frac{v_t}{v_t}=1$
Slope of velocity as a function of time is acceleration:
• Initial slope of velocity graph is 1.
‣ Initial acceleration is g.
$\frac{a(t)}{g}=\frac{a(0)}{g}=\frac{g}{g}=1$
• Final slope of velocity graph is 0.
‣ Final acceleration is 0.
$\frac{a(t)}{g}=\frac{a(\infty)}{g}=\frac{0}{g}=0$
(1,1)
initial slope = 1
$\frac{v(t)}{v_t}=1-e^{-\frac{t}{\tau}}$
v(t)/v$_t$ (dimensionless velocity ratio)
t/τ (dimensionless time ratio)
$\frac{a(t)}{g}=e^{-\frac{t}{\tau}}$
a(t)/g (dimensionless acceleration ratio)
t/τ (dimensionless time ratio)
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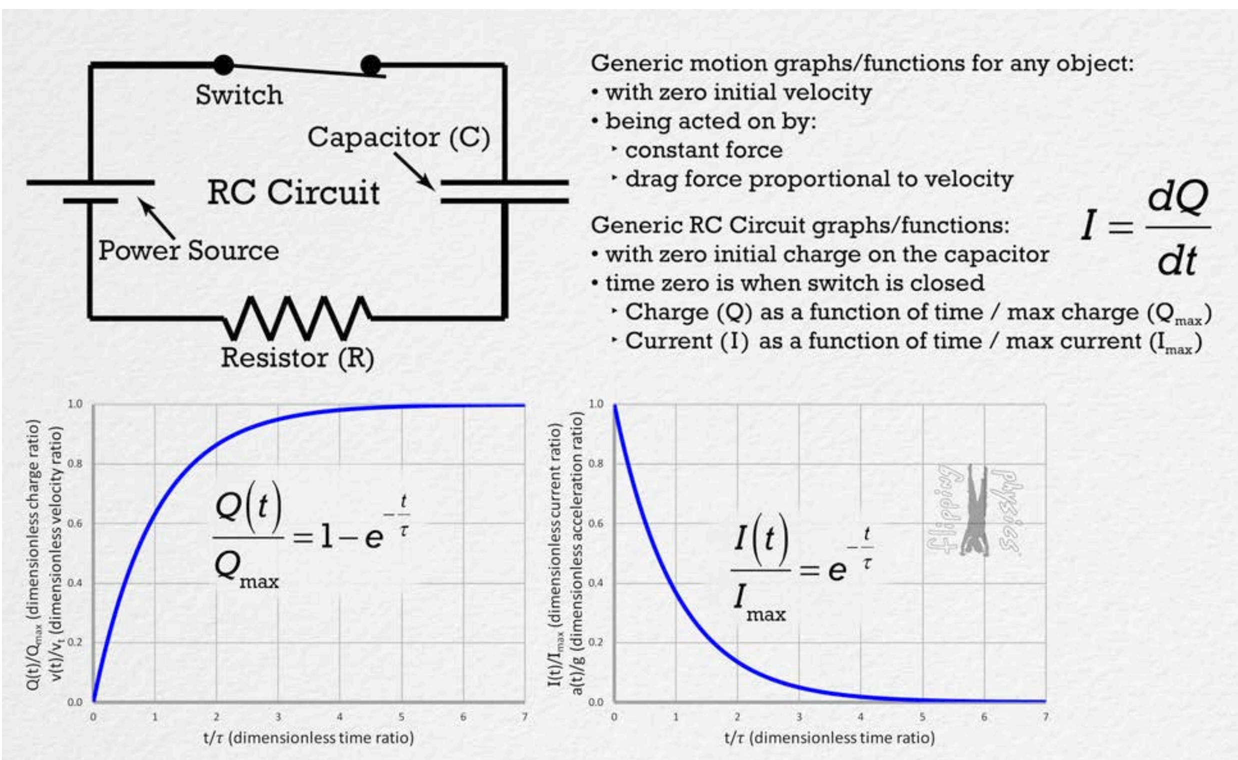
Switch
Capacitor (C)
RC Circuit
Power Source
Resistor (R)
Generic motion graphs/functions for any object:
• with zero initial velocity
• being acted on by:
‣ constant force
‣ drag force proportional to velocity
Generic RC Circuit graphs/functions:
• with zero initial charge on the capacitor
• time zero is when switch is closed
‣ Charge (Q) as a function of time / max charge (Q_{max})
‣ Current (I) as a function of time / max current (I_{max})
$I=\frac{dQ}{dt}$
$\frac{Q(t)}{Q_{max}}=1-e^{-\frac{t}{\tau}}$
Q(t)/Q$_{max}$ (dimensionless charge ratio)
v(t)/v$_t$ (dimensionless velocity ratio)
t/τ (dimensionless time ratio)
$\frac{I(t)}{I_{max}}=e^{-\frac{t}{\tau}}$
I(t)/I$_{max}$ (dimensionless current ratio)
a(t)/g (dimensionless acceleration ratio)
t/τ (dimensionless time ratio)
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