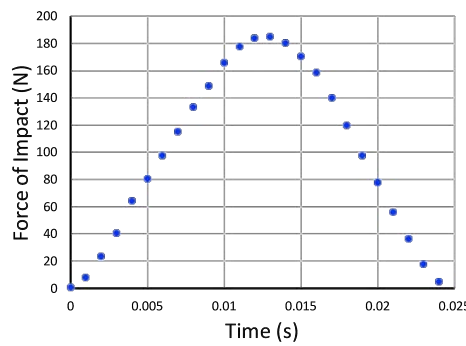
Flipping Physics Lecture Notes:

Demonstrating Calculus with a Ball and Force Platform

http://www.flippingphysics.com/calculus-demonstration.html

A 321 g rubber, playground ball is dropped from a height of 77.8 cm above a force platform. The data for the force of impact collected at 1000 data points per second as a function of time is shown. Please determine a bunch of stuff.

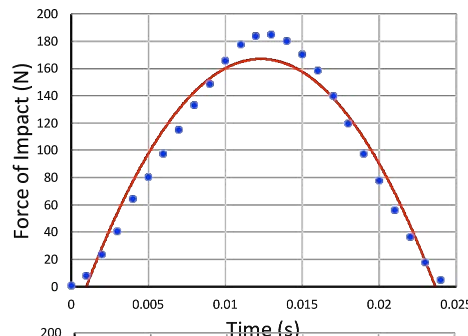


Add a best fit curve[1] to the data (see graph). The equation for the force of impact as a function of times then is:

$$F\left(t\right)=\left[-1.303\times10^{6}t^{2}+32130t-30.67\right]N$$

In order to make our calculations a little bit easier in this problem, we are going to adjust the force function to start at an initial time of zero. To do that we start by using the quadratic formula:

$$t=\frac{-b\pm\sqrt{b^{2}-4ac}}{2a}=\frac{-32130\pm\sqrt{32130^{2}-\left(4\right)\left(-1.303\times10^{6}\right)\left(-30.67\right)}}{\left(2\right)\left(-1.303\times10^{6}\right)}=0.00099468\text{sec } or \text{ } 0.023664\text{sec}$$



We have solved for the two times at which the force equals zero. Now, if we subtract 0.00099468 seconds from each of our time data points, we get a best fit force function which starts with zero force at time zero. Ee now have a new force as a function of time equation:

$$F\left(t\right)=\left[-1.303\times10^{6}t^{2}+29540t\right]N$$

Which we can set equal to zero:

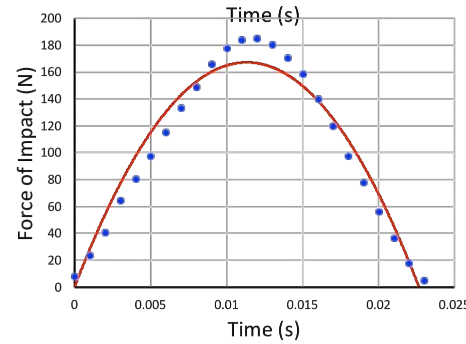
$$\Rightarrow 0=-1.303\times10^{6}t^{2}+29540t$$

And solve for time:

$$\Rightarrow 1.303\times10^{6}t=29540$$

$$\Rightarrow t=\frac{29540}{1.303\times10^{6}}=0.022671\text{sec}\approx22.7ms$$

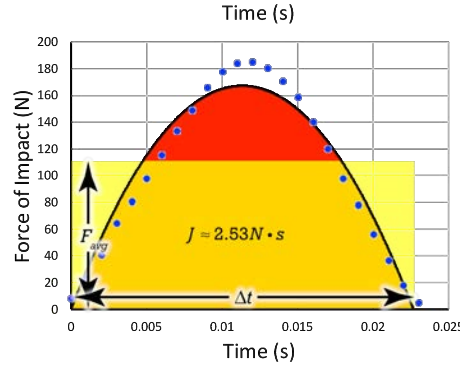
Which is the total time of the collision.



Solve for the impulse during the collision. Notice the initial time for the collision is at zero seconds, and the final time for the collision is at 0.022671 seconds.

$$J=\int_{t_i}^{t_f}F\,dt=\int_{0}^{0.022671}\left(-1.303\times10^{6}t^{2}+29540t\right)dt$$

$$\Rightarrow J=\left[\frac{-1.303\times10^{6}t^{3}}{3}+\frac{29540t^{2}}{2}\right]_{0}^{0.022671}$$



[1] I am fully aware that a polynomial of order 4 better fits these data. The raw data are available on my website. You are welcome to do these calculations with a polynomial of order 4 instead of the polynomial of order 2. 😬

$$\Rightarrow J = \frac{\left(-1.303\times10^{6}\right)\left(0.022671\right)^{3}}{3} + \frac{\left(29540\right)\left(0.022671\right)^{2}}{2} - 0 = 2.53041 \approx 2.53 N\bullet s$$

Use the impulse to determine the average force of impact during the collision:

$$J = F_{avg}\Delta t \Rightarrow F_{avg} = \frac{J}{\Delta t} = \frac{2.53041}{0.022671} = 111.61 \approx 112 N$$

Confirm the Impulse Approximation:

$$F_g = mg = \left(0.321\right)\left(9.81\right) = 3.14901 \approx 3.15 N \Rightarrow F_{impact} >> F_g$$

Use Newton's Second Law to solve for the acceleration of the center of mass of the ball during the collision:

$$\sum \vec{F} = m\vec{a} \Rightarrow \vec{a} = \frac{\sum \vec{F}}{m} = \frac{-1.303\times10^{6}t^{2} + 29540t}{0.321} \Rightarrow a\left(t\right) = \left[-4.0592\times10^{6}t^{2} + 92025t\right]\frac{m}{s^{2}}$$

Take the derivative of acceleration do find the jerk:

$$jerk = \frac{da}{dt} = \frac{d}{dt}\left(-4.0592\times10^{6}t^{2} + 92025t\right) \Rightarrow jerk\left(t\right) = \left[-8.1184\times10^{6}t + 92025\right]\frac{m}{s^{3}}$$

Set the jerk equal to zero to find the time at which the maximum force of impact occurs:

$$\Rightarrow -8.1184\times10^{6}t + 92025 = 0 \Rightarrow t = \frac{92025}{8.1184\times10^{6}} = 0.011335\,\text{sec} \approx 11.3 ms$$

Use that time to determine the maximum force of impact during the collision:

$$F\left(t\right) = \left[-1.303\times10^{6}t^{2} + 29540t\right]N$$

$$\Rightarrow F_{\max} = F\left(0.011335\right) = \left(-1.303\times10^{6}\right)\left(0.011335\right)^{2} + \left(29540\right)\left(0.011335\right) = 167.42 \approx 167N$$

Use an integral to determine the velocity of the center of mass of the ball as a function of time:

$$v\left(t\right) = \int a\,dt = \int\left(-4.0592\times10^{6}t^{2} + 92025t\right)dt = \frac{-4.0592\times10^{6}t^{3}}{3} + \frac{92025t^{2}}{2} + C$$

$$\Rightarrow v\left(t\right) = -1.3531\times10^{6}t^{3} + 46012.5t^{2} + C$$

And solve for the constant of integration by setting time equal to 0.011335 seconds because we know the velocity of the center of mass of the ball is zero at that point. Before this time the ball is moving down, after this time the ball is moving up, therefore, at this time, the ball must have a velocity of zero:

$$\Rightarrow v\left(0.011335\right) = 0 = \left(-1.3531\times10^{6}\right)\left(0.011335\right)^{3} + \left(46012.5\right)\left(0.011335\right)^{2} + C$$

$$\Rightarrow C = -3.9412 \approx -3.94\frac{m}{s} = v_i$$

We know have the equation for the velocity of the center of mass of the ball which it is colliding with the ground:

$$v\left(t\right) = \left[-1.3531\times10^{6}t^{3} + 46012.5t^{2} - 3.9412\right]\frac{m}{s}$$

The constant of integration is the initial velocity of the ball right before it strikes the ground.

We know the initial height of the ball, so we can check our velocity using *Conservation of Energy*. Set the initial point at the top where the ball starts, the final point right before the ball strikes the ground, and set the zero line at the center of mass of the ball right before the ball strikes the ground:

$$ME_i = ME_f \Rightarrow mgh_i = \frac{1}{2}mv_f^{\ 2} \Rightarrow gh_i = \frac{1}{2}v_f^{\ 2} \Rightarrow v_f = \sqrt{2gh_i} = \sqrt{(2)(9.81)(0.778)} = -3.90696 \approx -3.91\frac{m}{s}$$

Clearly, that velocity is very close to what we got for the initial velocity of the collision!

Another integral will result in the position of the center of mass of the ball during the collision:

$$x(t) = \int v\,dt = \int\left(-1.3531\times10^6 t^3 + 46012.5t^2 - 3.9334\right)dt = \frac{-1.3531\times10^6 t^4}{4} + \frac{46012.5t^3}{3} - 3.9334t + C$$

The constant of integration is arbitrary here, it is simply the initial position of the center of mass of the ball during the collision. Let's simply chose that initial position to be zero. So, the equation is:

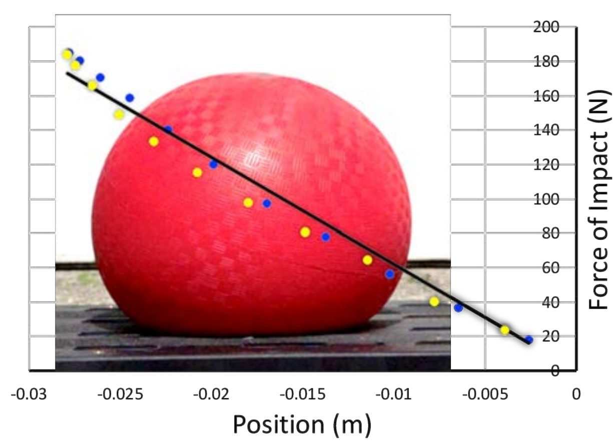
$$let\ x(0) = 0 = C \Rightarrow x(t) = \left[-3.3828\times10^5 t^4 + 15338t^3 - 3.9412t\right]m$$

Use that equation to solve for the maximum displacement of the center of mass of the ball during the collision:

$$x(0.011335) = \left(-3.3828\times10^5\right)(0.011335)^4 + (15338)(0.011335)^3 - (3.9412)(0.011335)$$

$$x_{\max} = -0.027920m \approx -2.79cm$$

Using the equations for force of impact as a function of time and position of the center of mass of the ball as a function of time, we can create a graph of force vs. time.



But the question is, what does this graph mean? First off, we have to translate the best fit line equation:

$$F = -6204x \ \&\ F_{spring} = -kx$$

In other words, the rubber, playground ball behaves like a spring, especially at small compression. And the spring constant of the ball is:

$$k = 6204\frac{N}{m}$$

But what does that mean? To understand that, let's do some conversions:

$$k = 6204\frac{N}{m}\left(\frac{1m}{100cm}\right) = 62.04 \approx 62.0\frac{N}{cm} \ \&\ k = 6204\frac{N}{m}\left(\frac{1lb}{4.448N}\right)\left(\frac{1m}{39.37in}\right) = 35.428 \approx 35.4\frac{lb}{in}$$

In other words, it takes roughly 62 N to compress the ball such that its center of mass moves 1 cm. And it takes roughly 35 lb to compress the ball such that its center of mass moves 1 inch. That seems completely reasonable to me.

To review, we collected the force of impact acting on a dropped ball. We used that to find the following information about the collision:

- Force of impact as a function of time
- Total time during the collision
- Impulse acting on the ball
- Average force of impact
- Acceleration of the center of mass
- Jerk as a function of time
- Time for maximum force of impact
- Value for the maximum force of impact
- Velocity as a function of time
- Initial velocity during the collision
- Position as a function of time
- Maximum displacement of the center of mass
- Spring constant of the rubber, playground ball.

That is the POWER and JOY of calculus!