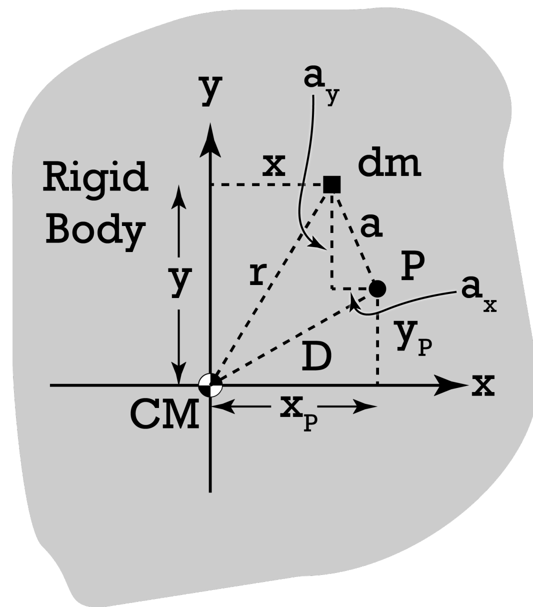
Flipping Physics Lecture Notes:

Parallel-Axis Theorem Derivation
http://www.flippingphysics.com/parallel-axis-Theorem.html

The Parallel-Axis Theorem is a convenient way to determine the moment of inertia or rotational inertia of a rigid body about an axis which is parallel to an axis which runs through the center of mass of the object. The Parallel-Axis Theorem states that the rotational inertia about any axis parallel to and a distance D away from the axis which passes through the center of mass of the object equals the rotational inertia about the center of mass of the object plus capital M, the total mass of the object, multiplied by the square of capital D, which is defined as the linear distance between the two parallel axes:

$$I = I_{CM} + MD^2$$

In order to prove the parallel-axis theorem, let's begin with a rigid object with shape and let's determine its rotational inertia around an axis which is perpendicular to the page and passes through point P, a point which is not at its center of mass. Place the origin of our coordinate system at the center of mass (CM) of the object. Identify an infinitesimally small piece of the object dm which is located at (x,y). Point P is located at (x_p,y_p).



The rotational inertia of the rigid object about its center of mass and about the axis which passes through point P are:

$$I_{CM} = \int r^2\,dm \Rightarrow I_P = \int a^2\,dm$$

Where r is defined as the distance from the origin (and the center of mass) to dm, and a is defined as the distance from dm to point P and a has components in the x and y direction such that, according to the Pythagorean Theorem:

$$a^2 = (a_x)^2 + (a_y)^2$$

And the distances for P and dm from the origin are related as such:

$$x_P = x + a_x \Rightarrow a_x = x_P - x \;\&\; y = y_P + a_y \Rightarrow a_y = y - y_P$$

Therefore, we can get a new value for a^2 substitute that back into the rotational inertia equation:

$$\Rightarrow a^2 = (x_P - x)^2 + (y - y_P)^2 \Rightarrow I_P = \int \left[(x_P - x)^2 + (y - y_P)^2\right] dm$$

And we can do some algebra:

$$\Rightarrow I_P = \int \left[\left(x_P^2 - 2x_P x + x^2\right) + \left(y^2 - 2y_P y + y_P^2\right)\right] dm$$

$$\Rightarrow I_P = \int \left[\left(x_P^2 + y_P^2\right) - 2x_P x - 2y_P y + \left(x^2 + y^2\right)\right] dm$$

$$\Rightarrow I_P = \int \left(x_P^2 + y_P^2\right) dm - \int 2x_P x\,dm - \int 2y_P y\,dm + \int \left(x^2 + y^2\right) dm$$

It is important to realize, because dm is in a variable location, however, the center of mass and point P are in fixed locations, that x_p and y_p are constants, however, x and y are variables. Therefore:

$$\Rightarrow I_P = \left(x_P^2 + y_P^2\right)\int dm - 2x_P \int x\,dm - 2y_P \int y\,dm + \int \left(x^2 + y^2\right) dm$$

And recall that the x and y centers of mass of our object are located at the origin, therefore both the x and y centers of mass of the object are zero:

$$x_{CM} = \frac{1}{m_{total}} \int x \, \mathrm{d}m = 0 \ \& \ y_{CM} = \frac{1}{m_{total}} \int y \, \mathrm{d}m = 0$$

Therefore, we can also conclude that:

$$\int x \, \mathrm{d}m = \int y \, \mathrm{d}m = 0 \Rightarrow I_P = \left(x_P^2 + y_P^2\right) \int \mathrm{d}m + \int \left(x^2 + y^2\right) \mathrm{d}m$$

Looking at just the first term:

$$\left(x_P^2 + y_P^2\right) \int \mathrm{d}m = D^2 \int \mathrm{d}m = D^2 M = MD^2$$

where $D^2 = x_P^2 + y_P^2$ & M is the total mass of the object

And just the second term:

$$\int \left(x^2 + y^2\right) \mathrm{d}m = \int r^2 \, \mathrm{d}m = I_{CM} \text{ where } r^2 = x^2 + y^2$$

Therefore, substituting back into the equation for moment of inertia:

$$\Rightarrow I_P = MD^2 + I_{CM} \Rightarrow I_P = I_{CM} + MD^2$$

We have derived the Parallel-Axis Theorem!