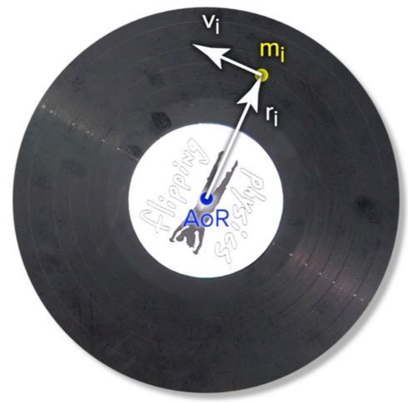
Flipping Physics Lecture Notes:

Angular Momentum of a Rigid Body Derivation

http://www.flippingphysics.com/angular-momentum-rigid-body.html



Let's say we have a rigid object with shape, or a rigid body, rotating about a fixed axis. We are going to derive the angular momentum of this rigid body. Each particle of the rigid body rotates in the xy plane of the screen with the same angular speed ω. The magnitude of the angular momentum of a single particle of the rigid object about the vertical z axis which has mass m_i that is r_i from the axis of rotation and is moving with a linear velocity v_i is:

$$L = rmv\sin\theta \Rightarrow L_i = r_i m_i v_i \sin(90°) = r_i m_i v_i$$

Because:

$$v_t = r\omega \Rightarrow v_i = r_i\omega_i \Rightarrow L_i = r_i m_i (r_i\omega_i) \Rightarrow m_i r_i^2 \omega_i$$

The direction of this angular momentum is the same direction as the angular velocity of the particle which, according to the right-hand rule is out of the screen or the positive z direction.

Now we can find the angular momentum of the entire rigid body by taking the sum of the angular momenta over all the particles in the object:

$$L_z = \sum_i L_i = \sum_i m_i r_i^2 \omega_i = \left(\sum_i m_i r_i^2\right)\omega \Rightarrow L_z = I\omega \text{ where } I = \sum_i m_i r_i^2$$

We have just derived that the angular momentum of a rigid object around its axis of rotation equals the rotational inertia of the object about its axis of rotation times its angular velocity.

We can take the derivative of this equation with respect to time: $\frac{dL_z}{dt} = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = I\alpha$

We also know the rotational form of Newton's Second Law in terms of angular momentum:

$$\sum \tau_{external} = \frac{dL_z}{dt} \Rightarrow \sum \tau_{external} = I\alpha$$

And we have derived the rotational form of Newton's Second Law for a rigid body.