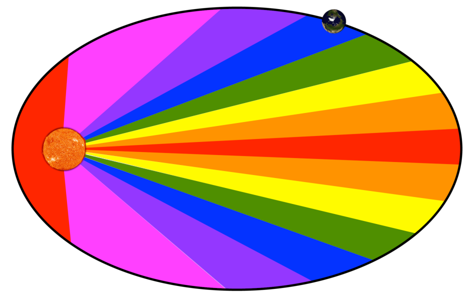
Flipping Physics Lecture Notes:

Kepler's Second Law Derivation

http://www.flippingphysics.com/kepler-second-derivation.html

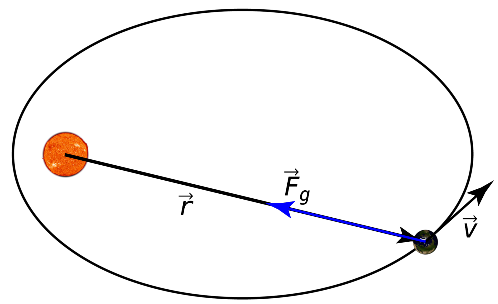
Kepler's Second Law of Planetary Motion:
"Each planet moves such that an imaginary line drawn between the sun and the planet sweeps out equal areas in equal time periods."



During the orbit of the planet, the force of gravity acting on the planet is always directed along the radius from the planet toward the Sun. The Sun is also called the primary. That force of gravity is the only force acting on the planet. Therefore, the net external torque acting on the planet with the axis of rotation at the center of the primary is:

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \sum_{\substack{\text{on the planet}\\ \text{AOR Sun}}} \vec{\tau} = \vec{\tau}_{F_g} = \frac{d\vec{L}_{\text{planet}}}{dt}$$

And the torque caused by the force of gravity is equal to zero because the cross product of the r position vector and the force of gravity equals zero. (The area of the parallelogram created by the two vectors equals zero.) Therefore, the angular momentum of the planet about the axis of rotation of the sun is conserved.



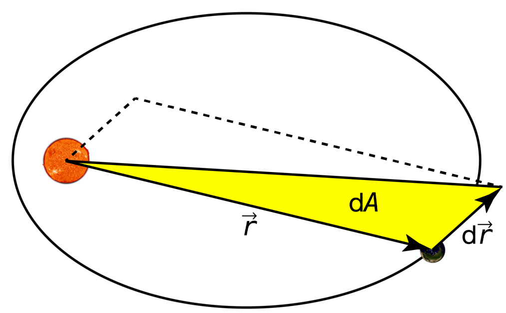
$$\vec{\tau}_{F_g} = \vec{r} \times \vec{F}_g = 0 \Rightarrow \frac{d\vec{L}_{\text{planet}}}{dt} = 0 \Rightarrow \vec{L}_{\text{planet}} = \text{constant}$$

And the angular momentum of the planet is:

$$\vec{L}_{\text{planet}} = \vec{r} \times \vec{p} = \vec{r} \times m_{\text{planet}}\vec{v} = m_{\text{planet}}\left(\vec{r} \times \vec{v}\right) = \text{constant}$$

And we can rearrange that equation:

$$\Rightarrow \vec{r} \times \vec{v} = \frac{\vec{L}_{\text{planet}}}{m_{\text{planet}}} = \text{constant}$$



We already showed that the angular momentum of the planet is constant, and we know the mass of the planet is constant, therefore, we now know that the cross product of the r position vector and velocity of the planet is constant. This will be helpful a bit later in this derivation.

And now we discuss "equal areas in equal time periods".
Over an infinitesimally small time period dt, the planet has moved an infinitesimally small displacement d**r**:

$$\vec{v} = \frac{d\vec{r}}{dt} \Rightarrow d\vec{r} = \vec{v}dt$$

The area swept out over the time period dt is d**A** and equals half the area of the cross product of the r vector and d**r**:

$$d\vec{A} = \frac{1}{2}\left(\vec{r} \times d\vec{r}\right) = \frac{1}{2}\left(\vec{r} \times \vec{v}dt\right) = \frac{1}{2}\left(\vec{r} \times \vec{v}\right)dt \Rightarrow \frac{d\vec{A}}{dt} = \frac{1}{2}\left(\vec{r} \times \vec{v}\right) = \text{constant}$$

As we proved earlier, the cross product of the r position vector and velocity of the planet is constant; therefore, the derivative of the area swept out by the planet with respect to time is constant. That means the area swept out by the planet over equal time periods will be the same. We just proved Kepler's Second Law of Planetary Motion!