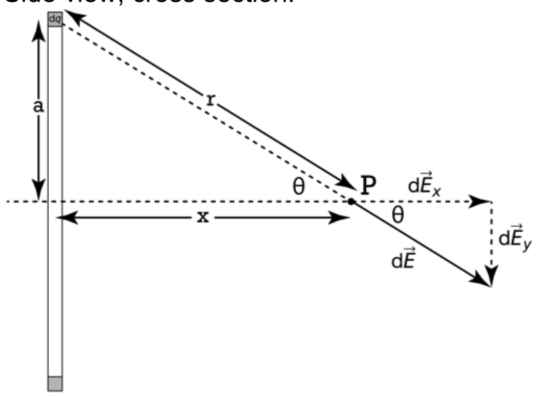
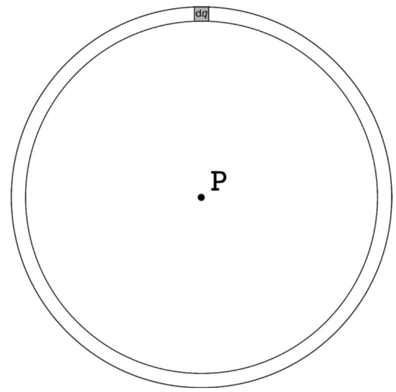
Flipping Physics Lecture Notes:
Thin Ring Electric Field
http://www.flippingphysics.com/thin-ring-electric-field.html

Let's determine the electric field caused by a uniformly charged ring of charge +Q, with radius a, at point P which is located on the axis of the ring a distance x from the center of the ring.

Side view, cross section:



Front view:



Knowns: a, x, Q

$$\vec{E} = k\int \frac{dq}{r^2}\hat{r} \Rightarrow d\vec{E} = k\left(\frac{dq}{r^2}\right)\hat{r} = k\left(\frac{dq}{a^2+x^2}\right)\hat{r}$$

However, all dE's in the vertical plane cancel out because there is an equal but opposite component of dE caused by the dq on the opposite side of the ring. In the figure that is dE_y.

$$d\vec{E}_x = d\vec{E}\cos\theta = k\left(\frac{dq}{a^2+x^2}\right)\hat{\imath}\cos\theta \Rightarrow \vec{E}_P = \int\left(\frac{k}{a^2+x^2}\hat{\imath}\cos\theta\right)dq$$

$$\& \quad \cos\theta = \frac{A}{H} = \frac{x}{r} = \frac{x}{\sqrt{a^2+x^2}} \Rightarrow \vec{E}_P = \int\left[\left(\frac{k}{a^2+x^2}\right)\left(\frac{x}{\sqrt{a^2+x^2}}\right)\hat{\imath}\right]dq$$

Note: $\left(a^2+x^2\right)\left(a^2+x^2\right)^{\frac{1}{2}} = \left(a^2+x^2\right)^{\frac{2}{2}}\left(a^2+x^2\right)^{\frac{1}{2}} = \left(a^2+x^2\right)^{\frac{3}{2}}$

$$\Rightarrow \vec{E}_P = \int\left(\frac{kx}{\left(a^2+x^2\right)^{\frac{3}{2}}}\right)\hat{\imath}\,dq = \left(\frac{kx}{\left(a^2+x^2\right)^{\frac{3}{2}}}\right)\hat{\imath}\int dq \Rightarrow \vec{E}_P = \left(\frac{kQx}{\left(a^2+x^2\right)^{\frac{3}{2}}}\right)\hat{\imath}$$

Note:

$$\text{if } x >> a \text{ then } a^2+x^2 \approx x^2 \;\&\; \vec{E}_P \approx \left(\frac{kQx}{\left(x^2\right)^{\frac{3}{2}}}\right)\hat{\imath} = \left(\frac{kQx}{x^3}\right)\hat{\imath} \qquad \Rightarrow \vec{E}_P \approx \left(\frac{kQ}{x^2}\right)\hat{\imath}$$

That's right, if you get far enough from the uniformly charged ring, it acts like a point particle!
If that sounds familiar, that is because this is true for all continuous charge distributions. If you get far enough away from them, that their own size is small by comparison to the distance, they all have electric fields which are similar to point particles.

Note:

$$\text{if } x << a \text{ then } a^2+x^2 \approx a^2 \;\&\; \vec{E}_P \approx \left(\frac{kQx}{\left(a^2\right)^{\frac{3}{2}}}\right)\hat{\imath} \Rightarrow \vec{E}_P \approx \left(\frac{kQx}{a^3}\right)\hat{\imath}$$

And if we use a negative charge, then the force is to the left or towards the center of the ring:

$$\sum F_x = -F_e = ma_x \Rightarrow -qE = -q\left(\frac{kQx}{a^3}\right) = ma_x \Rightarrow a_x = -\left(\frac{kqQ}{ma^3}\right)x$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{kqQ}{ma^3}\right)x \ \ \& \ \ \frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \omega = \sqrt{\frac{kqQ}{ma^3}} = \frac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\frac{ma^3}{kqQ}}$$

$$x(t) = A\cos(\omega t + \phi) \Rightarrow x_{max} = A$$

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi) \Rightarrow v_{max} = A\omega$$

$$a(t) = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \phi) \Rightarrow a_{max} = A\omega^2$$

That's right, the negative charge will move in simple harmonic motion about the center of the ring.

A bonus graph from Carl Hansen: (Thank you Carl Hansen!)

This following graph shows the equation we derived for the electric field along the axis of a thin ring of a uniform charge distribution, of radius a = 1m, and charge Q = +1μC. The plot shows the linear profile needed for simple harmonic motion close to the origin, and the inverse square law far away. The maximum electric field occurs around x = 0.7m, as a turning point to transition between the two trends.