Flipping Physics Lecture Notes:
Continuous Charge Distributions
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-continuous-charge-distributions.html

Continuous Charge Distribution: A charge that is not a point charge. In other words, a charge which has shape and continuous charge distributed throughout the object.

In order to find the electric field which exists around a continuous charge distribution, we can use Coulomb's Law and the equation definition of an electric field. We consider the charged object to be made up of an infinite number of infinitesimally small point charges dq and add up the infinite number of electric fields via superposition. It's an integral.

$$\vec{F}_e = k\frac{(q_1)(q_2)}{r^2}\hat{r} \;\; \& \;\; \vec{E} = \frac{\vec{F}_e}{q} \Rightarrow \vec{E}_{\text{point charge}} = k\frac{\frac{(q)(Q)}{r^2}}{q}\hat{r} = \frac{kQ}{r^2}\hat{r}$$

$$\Rightarrow d\vec{E} = \frac{k(\mathrm{d}q)}{r^2}\hat{r} \Rightarrow \int d\vec{E} = \int \frac{k(\mathrm{d}q)}{r^2}\hat{r} \Rightarrow \vec{E}_{\text{continuous charge distribution}} = k\int \frac{\mathrm{d}q}{r^2}\hat{r}$$

Realize that, for AP Physics C: Electricity and Magnetism, students are only expected to be able to use this equation to determine electric fields around continuous charge distributions with high symmetry. The specific examples students are responsible for are:

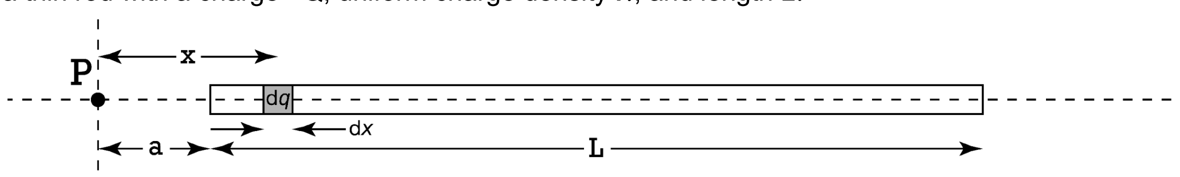
- An infinitely long, uniformly charged wire or cylinder at a distance from its central axis
- A thin ring of charge at a location along the axis of the ring
- A semicircular arc or part of a semicircular arc at its center
- A finite wire or line of charges at a distance that is collinear with the line of charge or at a location along its perpendicular bisector.

Quick review of charge densities:

linear charge density, $\lambda = \frac{Q}{L}$ in $\frac{C}{m}$ & surface charge density, $\sigma = \frac{Q}{A}$ in $\frac{C}{m^2}$

& volumetric charge density, $\rho = \frac{Q}{V}$ in $\frac{C}{m^3}$

Let's do an example. Let's determine the electric field at point P, which is located a distance "a" to the left of a thin rod with a charge +Q, uniform charge density λ, and length L.



Notice that, if we were to place a positive point charge at point P, it would experience a force to the left from every dq or every infinitesimally small part of the wire. Therefore, we already know the direction of the electric field at point P, it will be to the left or in the negative "i" direction. Now let's solve for the electric field:

$$\vec{E} = k\int \frac{\mathrm{d}q}{r^2}\hat{r} \Rightarrow \vec{E} = k\int \frac{\mathrm{d}q}{x^2}\left(-\hat{\imath}\right) \;\; \& \;\; \lambda = \frac{Q}{L} = \frac{\mathrm{d}q}{\mathrm{d}x} \Rightarrow \mathrm{d}q = \lambda \mathrm{d}x \;\; \& \;\; Q = \lambda L$$

$$\Rightarrow \vec{E} = -k\hat{\imath}\int \frac{\lambda \mathrm{d}x}{x^2} = -k\lambda\hat{\imath}\int_a^{a+L}\left(\frac{1}{x^2}\right)\mathrm{d}x = -k\lambda\hat{\imath}\int_a^{a+L}\left(x^{-2}\right)\mathrm{d}x$$

$$\Rightarrow \vec{E} = -k\lambda\hat{\imath}\left[\frac{x^{-1}}{-1}\right]_a^{a+L} = k\lambda\hat{\imath}\left[\frac{1}{x}\right]_a^{a+L} = k\lambda\hat{\imath}\left[\frac{1}{a+L} - \frac{1}{a}\right] = k\lambda\hat{\imath}\left[\frac{a-(a+L)}{a(a+L)}\right]$$

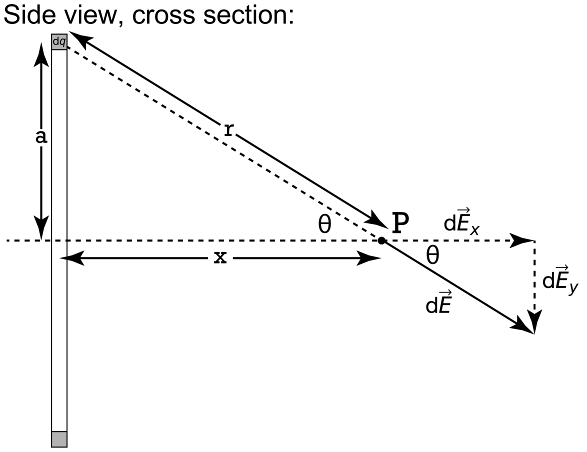
$$\Rightarrow \vec{E} = k\lambda\hat{\imath}\left[\frac{a-a-L}{a(a+L)}\right] = -\frac{k\lambda L}{a(a+L)}\hat{\imath} \Rightarrow \vec{E} = -\frac{kQ}{a(a+L)}\hat{\imath}$$

if $a >> L$ then $a + L \approx a$ & $\vec{E} = -\frac{kQ}{a^2}\hat{\imath}$

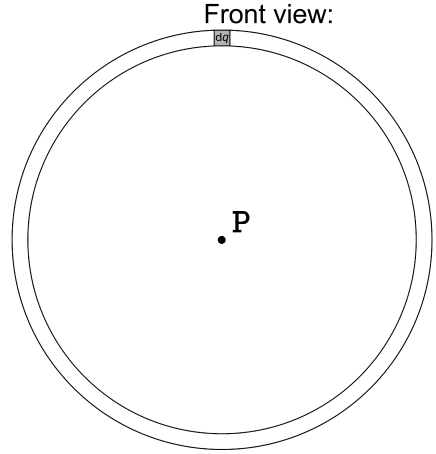
In other words, if we get far enough from the charged rod, it acts like a point charge. ☺

And, because they are so fun, another example! Let’s determine the electric field caused by a uniformly charged ring of charge +Q, with radius a, at point P which is located on the axis of the ring a distance x from the center of the ring.

Side view, cross section:



Front view:



Knowns: a, x, Q

$$\vec{E} = k\int \frac{dq}{r^2}\hat{r} \Rightarrow d\vec{E} = k\left(\frac{dq}{r^2}\right)\hat{r} = k\left(\frac{dq}{a^2+x^2}\right)\hat{r}$$

However, all dE’s in the vertical plane cancel out because there is an equal but opposite component of dE caused by the dq on the opposite side of the ring. In the figure that is dE_y.

$$d\vec{E}_x = d\vec{E}\cos\theta = k\left(\frac{dq}{a^2+x^2}\right)\hat{\imath}\cos\theta \Rightarrow \vec{E}_P = \int\left(\frac{k}{a^2+x^2}\hat{\imath}\cos\theta\right)dq$$

$$\& \quad \cos\theta = \frac{A}{H} = \frac{x}{r} = \frac{x}{\sqrt{a^2+x^2}} \Rightarrow \vec{E}_P = \int\left[\left(\frac{k}{a^2+x^2}\right)\left(\frac{x}{\sqrt{a^2+x^2}}\right)\hat{\imath}\right]dq$$

Note: $\left(a^2+x^2\right)\left(a^2+x^2\right)^{\frac{1}{2}} = \left(a^2+x^2\right)^{\frac{2}{2}}\left(a^2+x^2\right)^{\frac{1}{2}} = \left(a^2+x^2\right)^{\frac{3}{2}}$

$$\Rightarrow \vec{E}_P = \int\left(\frac{kx}{\left(a^2+x^2\right)^{\frac{3}{2}}}\right)\hat{\imath}\,dq = \left(\frac{kx}{\left(a^2+x^2\right)^{\frac{3}{2}}}\right)\hat{\imath}\int dq \Rightarrow \vec{E}_P = \left(\frac{kQx}{\left(a^2+x^2\right)^{\frac{3}{2}}}\right)\hat{\imath}$$

Note:

$$\text{if } x >> a \text{ then } a^2 + x^2 \approx x^2 \ \& \ \vec{E}_P \approx \left(\frac{kQx}{\left(x^2\right)^{\frac{3}{2}}}\right)\hat{\imath} = \left(\frac{kQx}{x^3}\right)\hat{\imath} \qquad \Rightarrow \vec{E}_P \approx \left(\frac{kQ}{x^2}\right)\hat{\imath}$$

That's right, if you get far enough from the uniformly charged ring, it acts like a point particle!
If that sounds familiar, that is because this is true for all continuous charge distributions. If you get far enough away from them, that their own size is small by comparison to the distance, they all have electric fields which are similar to point particles.

$$\text{if } x << a \text{ then } a^2 + x^2 \approx a^2 \ \& \ \vec{E}_P \approx \left(\frac{kQx}{\left(a^2\right)^{\frac{3}{2}}}\right)\hat{\imath} \Rightarrow \vec{E}_P \approx \left(\frac{kQx}{a^3}\right)\hat{\imath}$$

Note:
And if we use a negative charge, then the force is to the left or towards the center of the ring:

$$\sum F_x = -F_e = ma_x \Rightarrow -qE = -q\left(\frac{kQx}{a^3}\right) = ma_x \Rightarrow a_x = -\left(\frac{kqQ}{ma^3}\right)x$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{kqQ}{ma^3}\right)x \ \& \ \frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \omega = \sqrt{\frac{kqQ}{ma^3}} = \frac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\frac{ma^3}{kqQ}}$$

$$x(t) = A\cos(\omega t + \phi) \Rightarrow x_{max} = A$$

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi) \Rightarrow v_{max} = A\omega$$

$$a(t) = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \phi) \Rightarrow a_{max} = A\omega^2$$

That's right, the negative charge will move in simple harmonic motion about the center of the ring.

A bonus graph from Carl Hansen: (Thank you Carl Hansen!)

This following graph shows the equation we derived for the electric field along the axis of a thin ring of a uniform charge distribution, of radius a = 1m, and charge Q = +1μC. The plot shows the linear profile needed for simple harmonic motion close to the origin, and the inverse square law far away. The maximum electric field occurs around x = 0.7m, as a turning point to transition between the two trends.