Flipping Physics Lecture Notes:
Electric Potential
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-electric-potential.html

The electrostatic force is a conservative force, therefore:

$$F_x = -\frac{dU}{dx} \Rightarrow F_e = -\frac{dU_e}{dr} \Rightarrow dU_e = -\vec{F}_e \cdot d\vec{r} \Rightarrow \int dU_e = -\int \vec{F}_e \cdot d\vec{r}$$

$$\Rightarrow \Delta U_e = -\int \vec{F}_e \cdot d\vec{r} \;\;\&\;\; \vec{E} = \frac{\vec{F}_e}{q} \Rightarrow \vec{F}_e = q\vec{E} \Rightarrow \Delta U_e = -\int_A^B q\vec{E} \cdot d\vec{r}$$

$$\Rightarrow \Delta U_e = -q\int_A^B \vec{E} \cdot d\vec{r} = -q\int_A^B E\cos\theta\, dr$$

We have determined the change in electric potential energy experienced by a charged particle which has moved from point A to point B in an electric field. Notice, because the electrostatic force is a conservative force, this change in electric potential energy does not depend on the path taken from point A to point B.

To make a comparison to gravitational potential energy, if we lift an object vertically upward in a constant downward gravitational field, it will experience a positive change in gravitational potential energy.

The same is true for a positively charged object, if we move a positively charged object vertically upward in a constant downward electric field, it will experience a positive change in electric potential energy.

Notice the negative and the dot product in the equation. As we move a charge in a direction opposite the direction of the field, the direction of the displacement of the charge and the direction of the field are opposite to one another, therefore, the angle between those two directions is 180°, the cosine of 180° is negative one, which makes the change in potential energy of a positive charge positive.

$$\Rightarrow \Delta U_e = -q\int_A^B E\cos(180°)\, dr = -q\int_A^B E(-1)\, dr = q\int_A^B E\, dr$$

Next, we need to define **electric potential**. Just like we define the electric field in terms of the force experienced by a small, positive test charge, we define the electric potential in terms of the energy experienced by a small, positive test charge.

- $\vec{E} = \frac{\vec{F}_e}{q} \Rightarrow V = \frac{U_e}{q}$ in volts, $V = \frac{J}{C}$
- The symbol for electric potential is V. Yes, I know. The symbol for electric potential, V, is the same as the symbol for the units for electric potential, volts, V. It's not my fault.
- Electric potential is a **scalar** attribute of a **vector** electric field which does not depend on any electric charges which could be placed in that field.
    - The fact that electric potential is a scalar can be very helpful in this class.
    - This scalar can be either positive or negative for any given location.
    - Just like gravitational potential energy, we need to either assign a location where it equals zero, or follow a convention for assigning the zero potential location.
- Most often we work with **electric potential difference** not just electric potential. Electric potential difference is the difference in the electric potential between two points:[1]

[1] I know. I know. … Duh! … But it had to be said.

$$\Delta V = V_f - V_i = V_B - V_A \ \ \& \ \ V = \frac{U_e}{q}$$

$$\Rightarrow \Delta V = \frac{\Delta U_e}{q} = \frac{-q\int_A^B \vec{E}\cdot d\vec{r}}{q} = -\int_A^B \vec{E}\cdot d\vec{r}$$

- We will often set the initial electric potential, or electric potential at point A, equal to zero.
- Realize we can rearrange every integral to form a derivative:

$$\Delta V = -\int_A^B \vec{E}\cdot d\vec{r} \Rightarrow dV = -\vec{E}\cdot d\vec{r} \Rightarrow E_r = -\frac{dV}{dr}$$

Now that we have volts, the units for electric potential, it is important to realize the units for the electric field can be given in terms of volts as well.

$$\vec{E} = \frac{\vec{F}_e}{q} \text{ in } \frac{N}{C} = \left(\frac{N}{C}\right)\left(\frac{m}{m}\right) = \left(\frac{J}{C}\right)\left(\frac{1}{m}\right) = \frac{V}{m} \Rightarrow \frac{N}{C} = \frac{V}{m}$$

If a charge is moved from point A to point B via an external force, the external force does work on the charge, that changes the electric potential energy of the charge. And, as long as there is no change in the kinetic energy of the charge, that work equals the charge of the charge multiplied by the electric potential difference the charge went through: $W = q\Delta V$

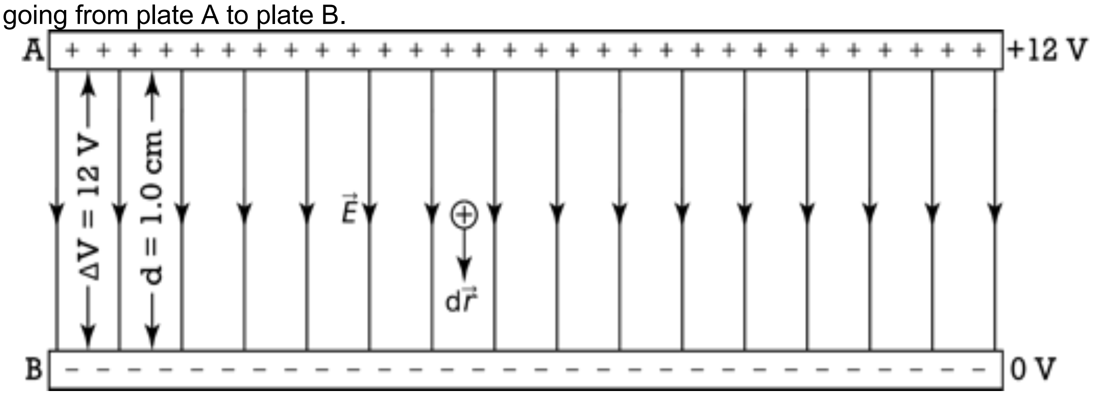
A unit of energy often used for very small amounts of energies, like one would use in atomic and nuclear physics, is the electron volt (eV). An electron volt is defined as the energy a charge-field system gains or losses when a charge of magnitude e (the elementary charge or the magnitude of the charge on an electron or proton) is moved through a potential difference of 1 V:

$$W = q\Delta V \Rightarrow W_{eV} = \left(1.6\times10^{-19}C\right)(1V) = 1.6\times10^{-19}C\cdot V \ \ \& \ \ C\cdot V = C\cdot\frac{J}{C} = J$$

$$\Rightarrow 1eV = 1.6\times10^{-19}J$$

I consider the electron volt to be a misnomer because it sounds like a unit of electric potential (volts), however, it is a unit of energy. It also refers to a positive amount of energy, even though the electron is negative. Be careful of that.

Let's say we have two, large, equal magnitude charged parallel plates, the top plate has a positive charge, and the bottom plate has a negative charge. We have shown the electric field is constant in this case and will be directed downward. Let's say the electric potential difference between the two plates is 12 volts and the distance between the two plates is 1.0 cm. Let's define the top plate as plate A, and the bottom plate as plate B. Let's start by determining the general equation for the electric potential difference when going from plate A to plate B.



$$\Delta V = -\int_A^B \vec{E} \cdot d\vec{r} = -\int_A^B E \cos\theta \, dr$$

$$\Rightarrow \Delta V = -\int_A^B E \cos(0°) \, dr = -E \int_A^B dr \Rightarrow \Delta V_{\text{constant E}} = -Ed$$

This is a good time to discuss the negative sign in the electric potential equation. In other words, a charge moving in the direction of the electric field will go through a negative potential difference and a charge moving opposite the direction of the electric field will go through a positive electric potential difference.

Add to the example: If we release a proton from the inside surface of plate A, what will the speed of the proton be right before it runs into plate B?

Set initial point at A and final point at B. Do not need a horizontal zero line because gravitational potential energy for subatomic particles is usually negligible and it is in this case. We do not actually know the electric potential energy initial or electric potential energy final; however, we do know the change in the electric potential energy. Also, the charge and mass of a proton are given in the Table of Information provided on the AP Physics C exam.

$$ME_i = ME_f \Rightarrow ME_A = ME_B \Rightarrow U_{\text{elec}_A} = KE_B + U_{\text{elec}_B}$$

$$\Rightarrow -KE_B = U_{\text{elec}_B} - U_{\text{elec}_A} = \Delta U_{\text{elec}} \Rightarrow -\frac{1}{2}mv_B^2 = q\Delta V$$

$$\& \ \ \Delta V = \frac{\Delta U_{\text{elec}}}{q} \Rightarrow \Delta U_{\text{elec}} = q\Delta V \ \ \& \ \ \Delta V_{A\to B} = -12V$$

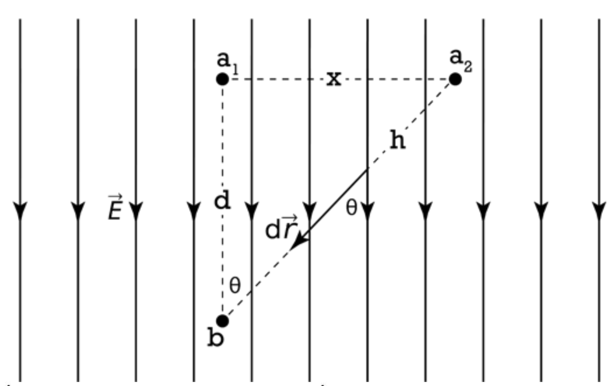
$$\Rightarrow v_B = \sqrt{-\frac{2q\Delta V}{m}} = \sqrt{-\frac{(2)\left(1.6\times 10^{-19}\right)(-12)}{1.67\times 10^{-27}}} = 47952\frac{m}{s} \approx 48\frac{km}{s}$$

$$v_B = 47952\frac{m}{s}\left(\frac{3600\ s}{1\ \text{hr}}\right)\left(\frac{1\ \text{mi}}{1609\ m}\right) = 107288\frac{\text{mi}}{\text{hr}} \approx 1.1\times 10^5\frac{\text{mi}}{\text{hr}}$$

Now let's look at determining the electric potential difference when moving at an angle relative to a uniform electric field. We already know the electric potential difference when moving from point a_1 to b:

$$\Delta V_{a_1\to b} = -Ed$$

Let's determine the electric potential difference when moving from point a_2 to b:



$$\Delta V_{a_2\to b} = -\int_{a_2}^b E\cdot dr = -\int_{a_2}^b E\cos\theta \, dr = -\int_{a_2}^b E\left(\frac{d}{h}\right)dr = -\left(\frac{Ed}{h}\right)\int_{a_2}^b dr$$

$$\Rightarrow \Delta V_{a_2 \to b} = -E\left(\frac{d}{h}\right)(h) = -Ed \Rightarrow \Delta V_{a_1 \to b} = \Delta V_{a_2 \to b}$$

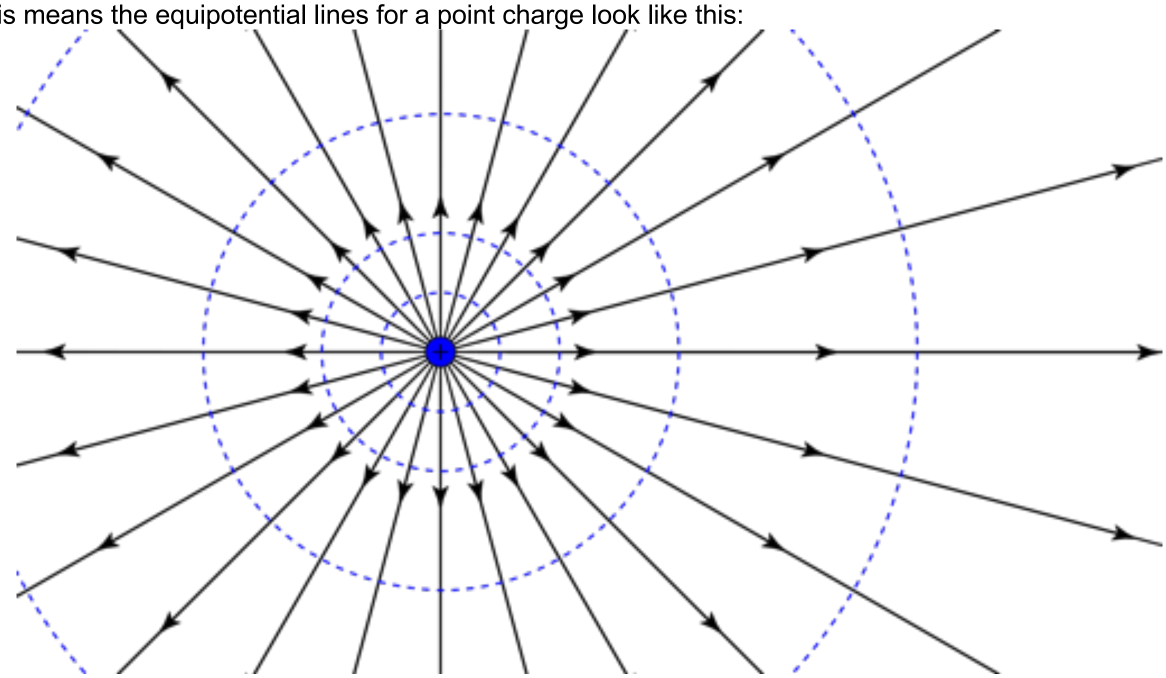
The electric potential difference is the same for both of these because points a_1 and a_2 have the same electric potential.

$$\Delta V_{a_2 \to a_1} = -\int_{a_2}^{a_1} E \cdot \mathrm{d}r = -E\int_{a_2}^{a_1} \cos\theta \, \mathrm{d}r = -E\int_{a_2}^{b} \cos(90°)\, dr = 0$$

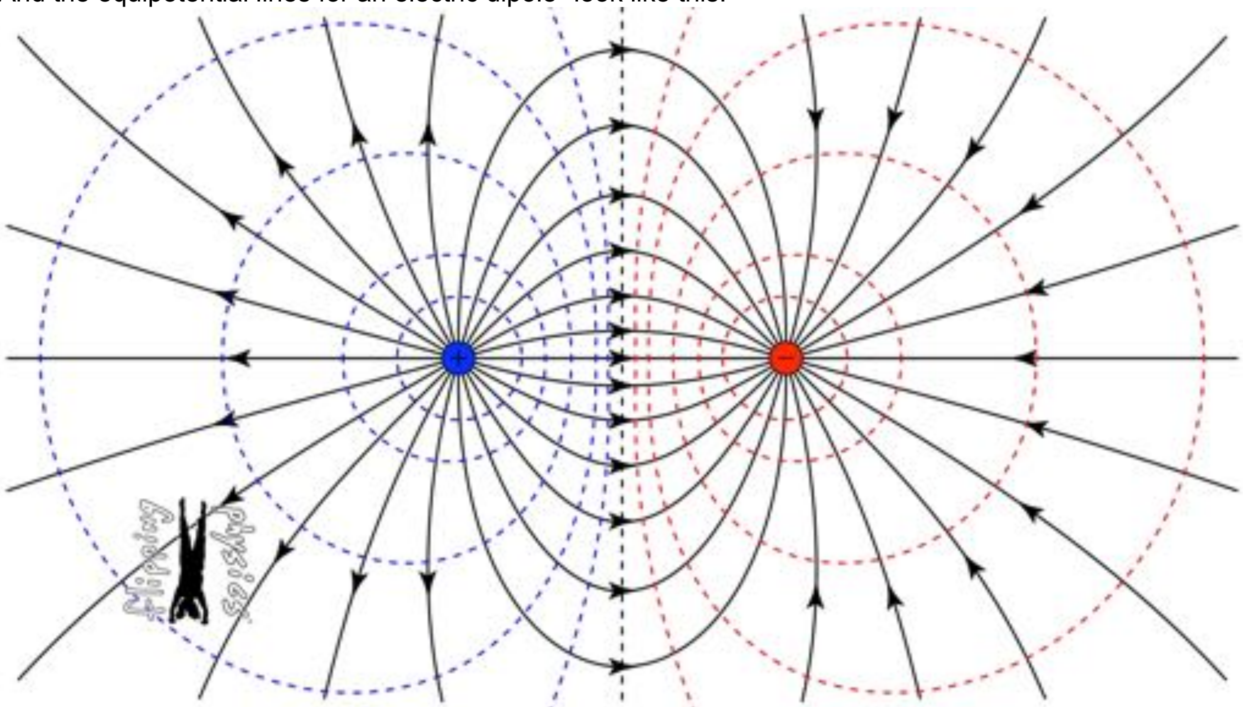
Points a_1 and a_2 are on an equipotential surface. An equipotential surface (or line):

- Has the same electric potential at every point on the surface (or line)
- Is always perpendicular to the electric field
    - Therefore, the electric field has no component along the equipotential line
- Equipotential lines are sometimes called isolines
- And it takes zero work to move a charged object along an equipotential surface
    - $W = q\Delta V \Rightarrow W_{\text{equipotential surface}} = q(0) = 0$

This means the equipotential lines for a point charge look like this:



And the equipotential lines for an electric dipole[2] look like this:



The equation for the electric potential which surrounds and is caused by a point charge is:

$$V_{\text{point charge}} = \frac{kq}{r}$$

This equation assigns our location of zero electric potential to be infinitely far away.

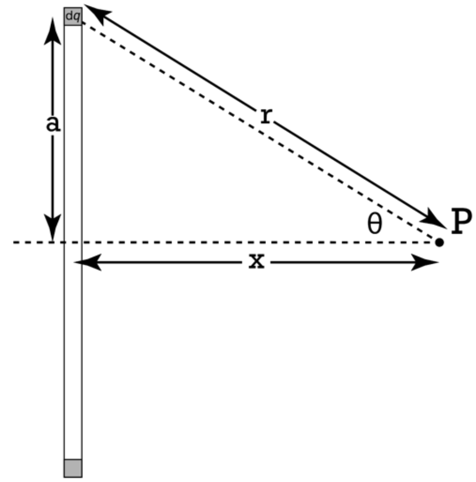
We can use the relationship between electric potential and electric potential energy to determine the electric potential energy which surrounds and is caused by a point charge:

$$V = \frac{U_{\text{elec}}}{q} \Rightarrow U_{\text{elec}} = qV \Rightarrow U_{\text{2 point charges}} = q_1\left(\frac{kq_2}{r}\right)$$

$$\Rightarrow U_{\text{2 point charges}} = \frac{kq_1q_2}{r}$$

[2] This is a simple example of an electric dipole which is a pair of electric charges of equal magnitude, but opposite sign separated by some typically small distance.

Realize, because electric potential and electric potential energy are scalar values, determining those values for multiple particles uses superposition. You just add all the values together.



In order to understand how useful it is that electric potential is a scalar and not a vector, let's revisit an example from before. Let's determine the electric potential caused by a uniformly charged, thin ring of charge +Q, with radius a, at point P, which is located on the axis of the ring a distance x from the center of the ring.

Because this is a continuous charge distribution, we need to break the uniformly charged thin ring of charge +Q into an infinite number of infinitesimally small charges, dQ.

$$V_{\text{point charge}} = \frac{kq}{r} \Rightarrow V_{\text{continuous charge distribution}} = \int \left(\frac{k}{r}\right) \mathrm{d}q = \frac{k}{r}\int \mathrm{d}q = \frac{kQ}{r} \Rightarrow V_P = \frac{kQ}{\sqrt{a^2 + x^2}}$$

And from there we can determine the electric field at point P.

$$E_r = -\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{kQ}{\sqrt{a^2 + x^2}}\right) = -kQ\frac{\mathrm{d}}{\mathrm{d}x}\left(a^2 + x^2\right)^{-\frac{1}{2}} = -kQ\left(-\frac{1}{2}\right)\left(a^2 + x^2\right)^{-\frac{3}{2}}(2x)$$

$$\Rightarrow E_P = \frac{kQx}{\left(a^2 + x^2\right)^{\frac{3}{2}}}$$

Notice that this derivation of the electric field at point P is much easier than deriving the electric field directly like we did before. Therefore, I would recommend that you remember that, for a continuous charge distribution, you can first determine the electric potential and then the electric field, and that is often easier than solving for the electric field directly.