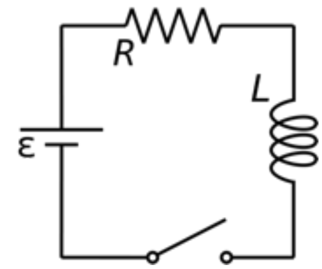
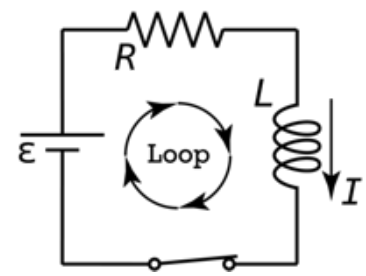
LR Circuits - Review for AP Physics C: Electricity and Magnetism

http://www.flippingphysics.com/apcem-lr-circuits.html



This LR circuit is a circuit with a battery, a resistor, an inductor, and a switch. Before time t = 0, the switch is open. At time t = 0, the switch is closed and remains closed. A few general things to realize:

- The initial current in the circuit, at time t = 0, is zero.
- The inductor opposes the change in current in the circuit which is what causes the current to change from its initial current of zero to its final steady state current.
- After a long time, the inductor behaves like any other ideal wire in a circuit and has zero resistance. In other words, after a long time the current has reached its maximum value and behaves as if the inductor is not there.



Let's determine equations for the limits. To do so, we use Kirchhoff's Loop Rule starting in the lower left-hand corner of the circuit:

$$\Delta V_{\text{Loop}} = 0 = \varepsilon - \Delta V_R - \Delta V_L = \varepsilon - IR - L\frac{dI}{dt}$$

We can use this equation to determine the remaining limits.

$@\ t_i = 0; I_i = 0$

$$\Rightarrow 0 = \varepsilon - L\frac{dI}{dt} \Rightarrow \varepsilon = L\frac{dI}{dt} \Rightarrow \frac{dI}{dt} = \frac{\varepsilon}{L} \Rightarrow \left(\frac{dI}{dt}\right)_{\text{initial}} = \frac{\varepsilon}{L}\ [\text{max value}]$$

$$@\ t_f \approx \infty; \left(\frac{dI}{dt}\right)_{\text{final}} = 0 \Rightarrow 0 = \varepsilon - IR \Rightarrow I_f = \frac{\varepsilon}{R}\ [\text{max value}]$$

And now we can derive the equation for current in this LR circuit as a function of time. Going back to the Kirchhoff's Loop Rule equation:

$$0 = \varepsilon - IR - L\frac{dI}{dt} \Rightarrow L\frac{dI}{dt} = \varepsilon - IR \Rightarrow \frac{L}{R}\frac{dI}{dt} = \frac{\varepsilon}{R} - I$$

$$\&\ \text{Let } u = \frac{\varepsilon}{R} - I \Rightarrow du = -dI \Rightarrow \frac{L}{R}\frac{-du}{dt} = u \Rightarrow \frac{du}{u} = -\frac{R}{L}dt$$

$$\Rightarrow \int \frac{du}{u} = \int -\frac{R}{L}dt \Rightarrow \int_{u_i}^{u_f} \frac{1}{u}du = -\frac{R}{L}\int_0^t dt \Rightarrow \ln u\Big|_{u_i}^{u_f} = -\frac{R}{L}t\Big|_0^t$$

$$\&\ \int \frac{dx}{x-a} = \ln|x-a| \Rightarrow \int \frac{du}{u} = \ln|u| = \ln u$$

- In this problem a = 0 and, because I varies from 0 to $\frac{\varepsilon}{R}$, u is always positive (or zero).

$$\Rightarrow \ln u_f - \ln u_i = \ln\left(\frac{u_f}{u_i}\right) = -\frac{R}{L}t \Rightarrow e^{\left(\ln\left(\frac{u_f}{u_i}\right)\right)} = e^{\left(-\frac{R}{L}t\right)} \Rightarrow \frac{u_f}{u_i} = e^{\left(-\frac{Rt}{L}\right)}$$

$$\Rightarrow u_f = u_i e^{\left(-\frac{Rt}{L}\right)}\ \&\ u_f = \frac{\varepsilon}{R} - I_f\ \&\ u_i = \frac{\varepsilon}{R} - I_i = \frac{\varepsilon}{R}$$

$$\Rightarrow \frac{\varepsilon}{R} - I_f = \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)} \Rightarrow -I_f = -\frac{\varepsilon}{R} + \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)} \Rightarrow I_f = \frac{\varepsilon}{R} - \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)}$$

$$\Rightarrow I(t) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{Rt}{L}\right)}\right) = I_{\max}\left(1 - e^{\left(-\frac{Rt}{L}\right)}\right)$$

Note, this fits our limits because:

- $I(0) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{R(0)}{L}\right)}\right) = \frac{\varepsilon}{R}\left(1 - e^{0}\right) = \frac{\varepsilon}{R}(1-1) = 0$
- $I(\infty) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{R(\infty)}{L}\right)}\right) = \frac{\varepsilon}{R}\left(1 - e^{-\infty}\right) = \frac{\varepsilon}{R}(1-0) = \frac{\varepsilon}{R}$

We can also determine the time rate of change of current as a function of time:

$$I_f = \frac{\varepsilon}{R} - \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)} \Rightarrow \frac{dI}{dt} = \frac{d}{dt}\left(\frac{\varepsilon}{R} - \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)}\right) = \left(-\frac{\varepsilon}{R}\right)\frac{d}{dt}e^{\left(-\frac{Rt}{L}\right)} = -\left(\frac{\varepsilon}{R}\right)\left(\frac{R}{L}\right)e^{\left(-\frac{Rt}{L}\right)}$$

$$\Rightarrow \frac{dI}{dt}(t) = \frac{\varepsilon}{L}e^{\left(-\frac{Rt}{L}\right)} = \left(\frac{dI}{dt}\right)_{\max} e^{\left(-\frac{Rt}{L}\right)} \quad \& \quad \frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

Again, this fits our limits because:

$$\frac{dI}{dt}(0) = \frac{\varepsilon}{L}e^{\left(-\frac{R(0)}{L}\right)} = \frac{\varepsilon}{L}e^{0} = \frac{\varepsilon}{L} \quad \& \quad \frac{dI}{dt}(\infty) = \frac{\varepsilon}{L}e^{\left(-\frac{R(\infty)}{L}\right)} = \frac{\varepsilon}{L}e^{-\infty} = 0$$

And we can determine the time constant, τ:

$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{t}{\tau}\right)}\right) \quad \& \quad \frac{dI}{dt}(t) = \frac{\varepsilon}{L}e^{\left(-\frac{t}{\tau}\right)} \Rightarrow \tau = \frac{L}{R}$$

$$\Rightarrow I(\tau) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{\tau}{\tau}\right)}\right) = \frac{\varepsilon}{R}\left(1 - e^{-1}\right) = \frac{\varepsilon}{R}(1 - 0.368) = 0.632\frac{\varepsilon}{R}$$

$$\Rightarrow \frac{dI}{dt}(\tau) = \frac{\varepsilon}{L}e^{\left(-\frac{\tau}{\tau}\right)} = \frac{\varepsilon}{L}e^{(-1)} = 0.368\frac{\varepsilon}{L} = (1 - 0.632)\frac{\varepsilon}{L}$$

And the graphs:

You can consider the derivative of current with respect to time to be the acceleration of moving objects. Amps are coulombs per second. So, amps per second are coulombs per second squared. The time rate of change of current is the rate at which the current is changing, just like acceleration is the rate at which velocity is changing.