Our current equation for kinetic energy is $KE = \dfrac{1}{2}mv^2$.

According to our kinetic energy equation, when an object is rotating around its center of mass, but its center of mass is not moving, the object has zero velocity; therefore, the object has zero kinetic energy. Kinetic energy is the energy of motion. Hopefully you recognize that an object rotating around its center of mass is moving and therefore must have kinetic energy, it is just not described by the equation above. We need to look at the kinetic energy of all the individual pieces of the rotating object.

The total kinetic energy of all the individual pieces of the rotating object is: $KE_{total} = \sum\limits_i KE_i$

This means the sum of all the kinetic energies of every piece that makes up the object. The letter "i" represents that the number goes from 1 to "i", the total number of pieces which make up the object. We can

substitute in the equation for kinetic energy: $KE_{total} = \sum\limits_i \dfrac{1}{2}m_i\left(v_i\right)^2$

The v_i represents the velocity of every piece which makes up the object. Notice this must be a tangential velocity, because the object is rotating and therefore every part of the object is moving in a circle.

Remember the equation that relates tangential velocity to angular velocity: $v_t = r\omega$

Therefore: $\Rightarrow KE_{total} = \sum\limits_i \dfrac{1}{2}m_i\left(r_i\omega_i\right)^2 = \sum\limits_i \dfrac{1}{2}m_i\left(r_i\right)^2\left(\omega_i\right)^2$

Assuming the object is a *rigid object with shape*, the angular velocity of every piece of the object will be the

same, therefore $\omega_i = \omega$

Notice the "r" in this equation is the distance each particle is from the axis of rotation, which is *not* the same for each piece.

We can isolate $\sum\limits_i m_i\left(r_i\right)^2$ in the equation: $\Rightarrow KE_{total} = \dfrac{1}{2}\left(\sum\limits_i m_i\left(r_i\right)^2\right)\omega^2$

We define $\sum\limits_i m_i\left(r_i\right)^2$ as the *Moment of Inertia* of the object and identify it with the symbol, capital I:

$$\boxed{I = \sum\limits_i m_i\left(r_i\right)^2}$$

We can substitute the object's moment of inertia back into the total kinetic energy equation to get the total

kinetic energy of a rotating object which is called *Rotational Kinetic Energy*: $\boxed{KE_{rotational} = \dfrac{1}{2}I\omega^2}$

The original kinetic energy then needs to be more specifically defined as *Translational Kinetic Energy*. In other words, the kinetic energy associated with the motion of the center of mass of the object moving from

one point in space to another point in space: $KE_{translational} = \dfrac{1}{2}mv^2$

To help understand what moment of inertia is, notice the moment of inertia takes the place of the inertial mass in the kinetic energy equation. That is why I like to think of moment of inertia as "rotational mass".

Remember inertial mass is a measure of the tendency of an object to resist acceleration. The more mass something has, the more it resists acceleration. This means that moment of inertia or "rotational mass" is a measure of the tendency of an object to resist *angular* acceleration. The more moment of inertia or "rotational mass" something has, the more it resists *angular* acceleration.

Two Eggs in an Egg Carton: A Moment of Inertia or "Rotational Mass" Example:
For this example, we are going to assume the egg carton has a small enough mass relative to the mass of the two eggs to be negligible.

Place two eggs in an egg carton, both near the middle like this:
Because there are two objects in the system, the moment of inertia will be:

$$I = \sum_i m_i \left(r_i \right)^2 \Rightarrow I = m_1 \left(r_1 \right)^2 + m_2 \left(r_2 \right)^2$$

Where m is the mass of each egg and r is the distance each egg is from the axis of rotation.

When we hold the egg carton in the middle and rotate it, it is relatively easy to rotate the system. In other words, because the eggs are close to the axis of rotation, the moment of inertia is low, and it is relatively easy to cause the eggs to angularly accelerate.

Now move the eggs so that they are on opposite ends of the egg carton like this:
The distance each egg is from the axis of rotation has been increased such that, now when we hold the egg carton in the middle and rotate it, it is more difficult to rotate the system. In other words, because we have increased the distance the eggs are from the axis of rotation, we have increased the moment of inertia or "rotational mass", and therefore it is more difficult to cause the eggs to angularly accelerate.
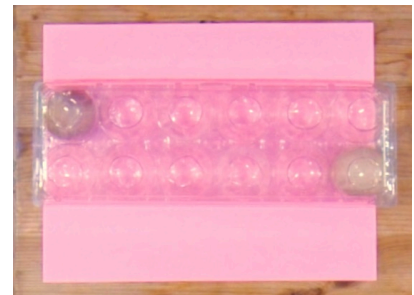
Realize we have not changed the mass of the system; we have only changed the locations of the masses. Increasing the distance the eggs are from the axis of rotation increases the moment of inertia or "rotational mass" of the system which makes it more difficult to angularly accelerate; however, the inertial mass of the system remains the same.

Lastly, notice how "r", the distance from the axis of rotation of each particle, is squared in the moment of inertia equation. This means the distance each particle is from the axis of rotation of the system has a much larger influence over the moment of inertia than the mass of each particle.

Example: Three 20.0-gram masses are 9.4 cm from an axis of rotation and rotating at 152 revolutions per minute. What is the moment of inertia of the three-object system? The strings holding the masses are of negligible mass.

$$I = \sum_i m_i \left(r_i \right)^2 \Rightarrow I = m_1 \left(r_1 \right)^2 + m_2 \left(r_2 \right)^2 + m_3 \left(r_3 \right)^2 = mr^2 + mr^2 + mr^2 = 3mr^2$$

All the masses and distances from the axis of rotation are the same:

$$m_1 = m_2 = m_3 = m \ \& \ r_1 = r_2 = r_3 = r$$

$$m = 20g \times \frac{1kg}{1000g} = 0.020kg \ \& \ r = 9.4cm \times \frac{1m}{100cm} = 0.094m$$

$$I = 3\left(0.020\right)\left(0.094\right)^2 = 0.00053016 \approx \boxed{5.3 \times 10^{-4} kg \cdot m^2}$$

Notice the moment of inertia of the system is independent of angular velocity.
This is the same as the mass of an object being independent of its velocity.

We can also determine the rotational kinetic energy of the system:

$$\omega = 152 \frac{rev}{min} \times \frac{1min}{60s} \times \frac{2\pi rad}{1rev} = 15.9174 \frac{rad}{s}$$

$$KE_{rot} = \frac{1}{2} I \omega^2 = \frac{1}{2}\left(0.00053016\right)\left(15.9174\right)^2 = 0.0671617 \approx 6.7 \times 10^{-2} J \times \frac{1000mJ}{1J} = \boxed{67mJ}$$
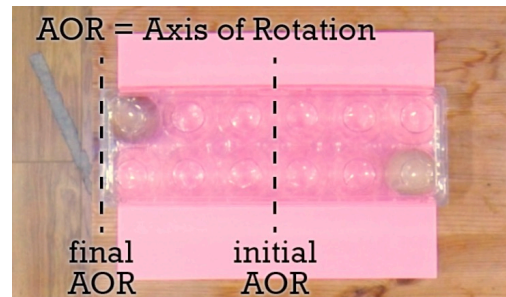
$$\frac{kg \cdot m^2 \cdot rad^2}{s^2} = \frac{kg \cdot m^2}{s^2} = \left(\frac{kg \cdot m}{s^2} \right) m = N \cdot m = J$$

Example: Two equal mass eggs are placed at either end in an egg carton of negligible mass. The egg carton is initially rotated about its middle. If the egg carton is now rotated about one end, what is the final moment of inertia of the eggs relative to their initial moment of inertia?

AOR = Axis of Rotation

final AOR   initial AOR

Two objects are in the system, so the moment of inertia equation has two expressions, one for each egg:

$$I = \sum_i m_i \left(r_i \right)^2 \Rightarrow I = m_1 \left(r_1 \right)^2 + m_2 \left(r_2 \right)^2$$

The mass of each egg is the same: $m_1 = m_2 = m$

Initially both eggs are roughly the same distance "r" from the middle of the egg carton, the initial axis of rotation: $r_{1i} = r_{2i} = r$

$$\Rightarrow I_i = m_1 \left(r_{1i} \right)^2 + m_2 \left(r_{2i} \right)^2 = mr^2 + mr^2 = 2mr^2$$

The final distances from the axis of rotation are: $r_{1f} \approx 0$ & $r_{2f} \approx 2r_i \approx 2r$

Therefore, the final moment of inertia is:

$$I_f = m_1 \left(r_{1f} \right)^2 + m_2 \left(r_{2f} \right)^2 = m\left(0 \right)^2 + m\left(2r \right)^2 = 4mr^2 = 2\left(2mr^2 \right)$$

And we can substitute the initial moment of inertia in for $2mr^2$, therefore:

$$\Rightarrow \boxed{I_f \approx 2I_i}$$

In other words, moving the axis of rotation from the middle of the egg carton to the end of the egg carton doubles the moment of inertia. That means it is twice as difficult to cause the two eggs to angularly accelerate around the axis of rotation.

Flipping Physics Lecture Notes:

Moments of Inertia of Rigid Objects with Shape

We are going to discuss six different equations for moments of inertia of rigid objects with constant density:

1. Thin, hollow cylinder about its long cylindrical axis.
2. Solid cylinder about its long cylindrical axis.
3. Thin, hollow sphere about is center of mass.
4. Solid sphere about its center of mass.
5. Thin rod about its center of mass.
6. Thin rod about one end.
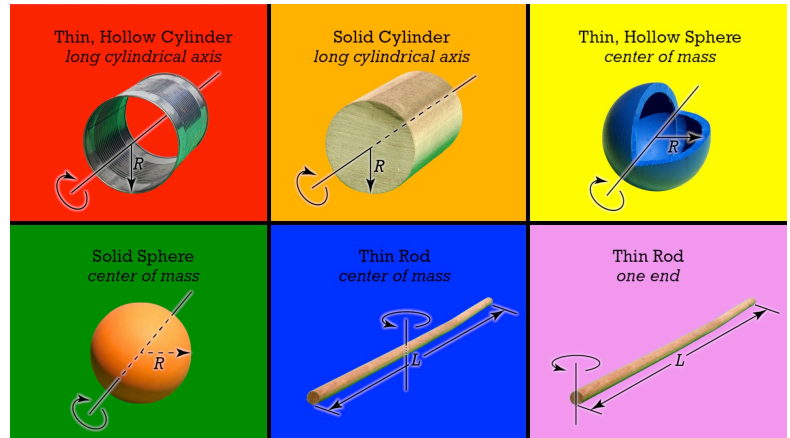
A "rigid" object will not easily change shape.
Please do not miss the fact that the density of these objects has to be constant.

In a typical calculus based physics class you would derive many of these moments of inertia, however, for this algebra based class, we will only discuss their relative moments of inertia. In my opinion it is not worth memorizing these equations, however, understanding why they have their relative values is *very* worthwhile.

Let's start with the last two: A thin rod about (a) its center of mass and (b) one end. First off realize a "thin" rod means the radius of the rod is very small relative to the length of the rod, so we consider the rod to essentially be one-dimensional. Both moments of inertia for the thin rod are a fraction times ML^2. Where "M" is the mass of the rod and "L" is the length of the rod.

The equation for the moment of inertia of a system of particles is: $I = \sum_i m_i \left(r_i \right)^2$.

The calculus version of moment of inertia of a rigid object with shape and constant density is: $I = \int r^2 \, dm$

The moment of inertia of the thin rod about its center of mass is: $I_{rod@center} = \dfrac{1}{12} ML^2$

Because some of the mass is farther from the axis of rotation when the rod is rotated about its end rather than about its center of mass and we are squaring the distance the pieces of the rigid object are from the axis of rotation in the moment of inertia equation, we would expect the moment of inertia about one end to be greater in value than about its center.[♥] $I_{rod@end} = \dfrac{1}{3} ML^2$ and $\dfrac{1}{3} > \dfrac{1}{12}$, so it works out.

The other four moments of inertia are a fraction times MR^2. Let's start with the thin, hollow cylinder about its long cylindrical axis. Again, the term "thin" here means the thickness of the hollow cylinder is very small relative to the radius of the cylinder, therefore we can consider the hollow cylinder to be essentially a two dimensional object where every piece of the thin, hollow cylinder is a distance R from the axis of rotation. That means every "r" value equals R, the radius of the hollow cylinder. Therefore the moment of inertia of a thin, hollow cylinder about its long cylindrical axis is MR^2. $I_{hollow\,cylinder} = MR^2$ (note the "fraction" here is 1.)

[♥] My apologies. The footnoted, 70-word sentence may be the longest I have ever written.

Because more of the mass of the solid cylinder is closer to the axis of rotation than for the thin, hollow cylinder, you would expect the fraction for the equation for the moment of inertia of a solid cylinder to be less than for a thin, hollow cylinder. $I_{solid\ cylinder} = \dfrac{1}{2}MR^2$ and $\dfrac{1}{2} < 1$, so it works out.

Notice neither of these two moments of inertia depend on the length of the cylinder. That means the equation for the moment of inertia of a solid disk is the same as for a solid cylinder. And the equation for the moment of inertia of a thin, hollow cylinder is the same as for a thin ring.

Next let's discuss the moment of inertia of a solid sphere about its center of mass. Compared to the solid cylinder, more of the solid sphere's mass is concentrated near its axis of rotation. Therefore, we would expect the fraction for the equation for the moment of inertia of a solid sphere to be less than for a solid cylinder. $I_{solid\ sphere} = \dfrac{2}{5}MR^2$ and $\dfrac{2}{5} < \dfrac{1}{2}$, so it works out.

Last is the moment of inertia of a thin, hollow sphere about its center of mass. Compared to the solid cylinder, a hollow sphere has a larger proportion of its mass located farther from the axis of rotation, so we would expect the fraction for the equation for the moment of inertia of a thin, hollow sphere about its center of mass to be more than for a solid cylinder. However, compared to the thin, hollow cylinder, a hollow sphere has a smaller proportion of its mass located farther from the axis of rotation, so we would expect the fraction for the equation for the moment of inertia of a thin, hollow sphere about its center of mass to be less than for a thin, hollow cylinder. $I_{hollow\ sphere} = \dfrac{2}{3}MR^2$ and $\dfrac{1}{2} < \dfrac{2}{3} < 1$, so it works out.

Again, please *do not memorize* these equations. Instead, **understand** why they have their relative values.

| Thin, Hollow Cylinder *long cylindrical axis* $I = MR^2$ | Solid Cylinder *long cylindrical axis* $I = \dfrac{1}{2}MR^2$ | Thin, Hollow Sphere *center of mass* $I = \dfrac{2}{3}MR^2$ |
| Solid Sphere *center of mass* $I = \dfrac{2}{5}MR^2$ | Thin Rod *center of mass* $I = \dfrac{1}{12}ML^2$ | Thin Rod *one end* $I = \dfrac{1}{3}ML^2$ |

Rigid Objects with Constant Density
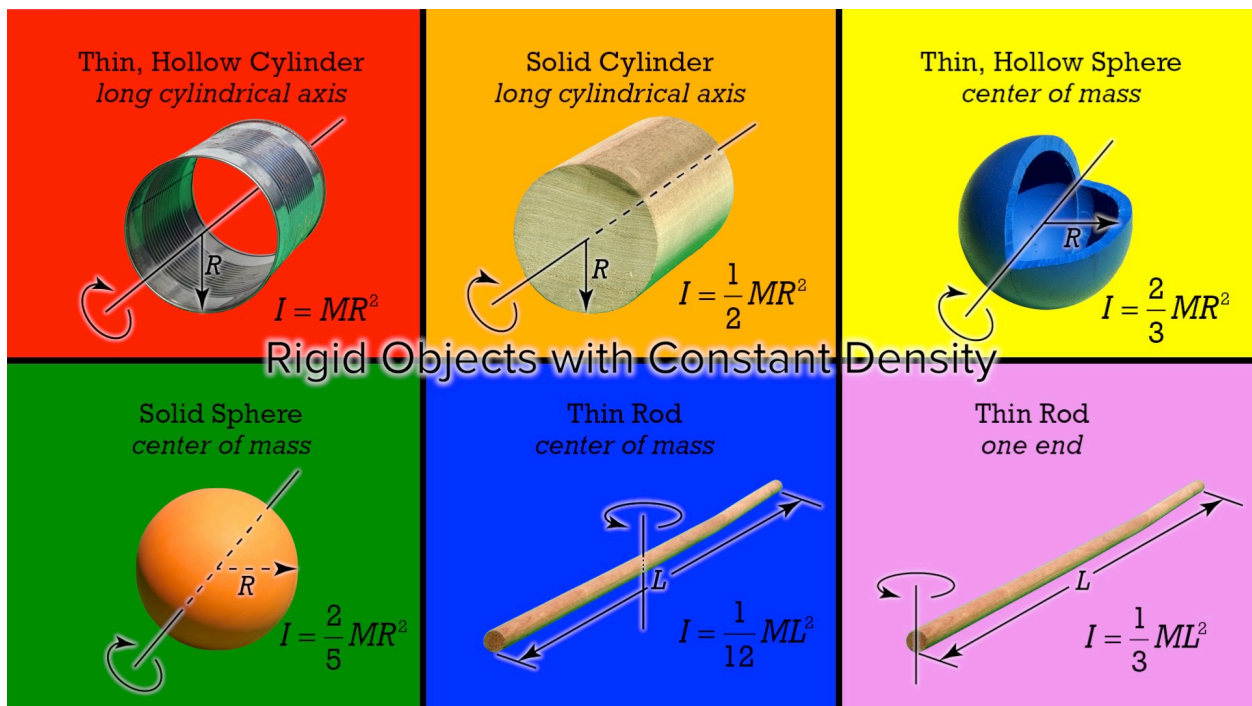
Two general types of motion:
- Translational Motion: The center of mass moves from one location to another location.
  - Caused by a net force.
  - Force is the ability to cause an acceleration of an object.
- Rotational Motion: The object moves in circular motion about its center of mass.
  - Caused by a net torque
  - Torque is the ability of a force to cause an angular acceleration of an object.

The equation for torque is $\boxed{\vec{\tau} = \vec{r}\vec{F}\sin\theta}$
- The symbol for torque is the lowercase Greek letter tau, τ.
- F is the force causing the torque.
- The equation is sometimes given as $\vec{\tau} = \vec{r}_{\perp}\vec{F}$
- $\vec{r}_{\perp}$ is called the "moment arm" or "lever arm" and $\vec{r}_{\perp} = \vec{r}\sin\theta$
- $\vec{r}$ is the position vector from the axis of rotation to where the force is applied to the object.
- θ is the angle between the direction of the force and the direction of $\vec{r}$.
- Torque is a vector, which means it has both magnitude and direction.
  - We will talk about direction in detail in the next lesson.

Everything you ever needed to know about torque, you already know, because you have opened many doors. I know that all seems a bit confusing, so let's walk through some example problems involving a door. When you approach a door you find the handle and the handle is always located far from the hinge. That is because the hinge is the axis of rotation. The distance from the axis of rotation to the location the force is applied (the handle) is the magnitude of the variable "r".
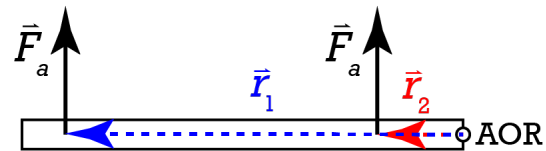
Let's start by assuming you are always pushing or pulling on the door at a 90 degree angle to the door:

$$\sin\left(90°\right)=1 \Rightarrow \vec{r}_{\perp} = \vec{r}\sin\theta = \vec{r}\sin\left(90\right) = \vec{r} \Rightarrow \vec{\tau} = \vec{r}\vec{F}$$

In this special case "r" times the force equals the torque. The handle is far from the axis of rotation so the torque associated with the force is large. When we push on the door with the same force near the axis of rotation, the "r" value is small and therefore, even with the same force, the torque is small, which means the ability to cause an angular acceleration of the door is small.

What if we push on the door at an angle which is not 90° like a 45° angle? Then, because of the shape of the sine curve, the torque associated with this force and "r" value will be reduced. In other words, because $\sin\left(90°\right)=1$, an angle of 90° between "r" and the force, will produce the largest torque.
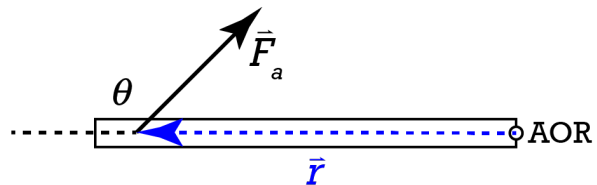
What if we push on the door at an angle of 0° or 180°?

$$\sin\left(0°\right) = \sin\left(180°\right) = 0 \Rightarrow \vec{\tau} = \vec{r}\vec{F}\sin\theta = \vec{r}\vec{F}\left(0\right) = 0$$

This results in zero torque or zero ability to cause an angular acceleration of the door.
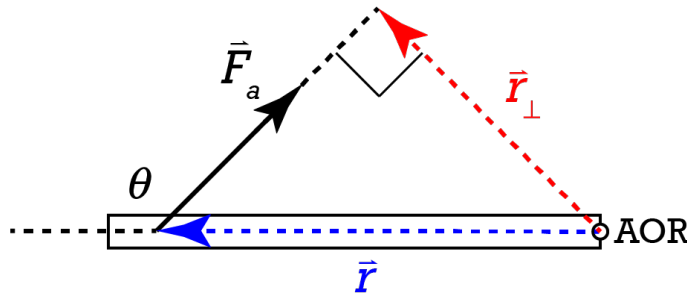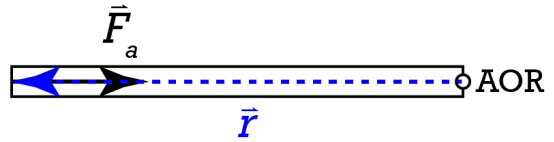
But what is $\vec{r}_{\perp}$, the "moment arm" or "lever arm"? Going back to pushing on the door at an angle.

Illustration. Notice "r" is the hypotenuse of a right triangle that has two sides which are the force and the moment arm. In this example, the moment arm has a smaller magnitude than "r". The only way the moment arm and "r" have the same value is if the angle is 90°. In other words, a 90° angle results in the largest torque, assuming the force and "r" value are the same. Notice also that if the angle is 0°, then the moment arm equals zero.

The units for torque are $\vec{\tau} = \vec{r}\vec{F}\sin\theta \Rightarrow m \cdot N \Rightarrow N \cdot m$.

Typically they are given as netwon meters instead of joules to differentiate torque from work and energy.

Example: To tighten a lag bolt, a 29 N force is applied at a 90° angle to a wrench 0.18 m from the center of a lag bolt. If the angle between the wrench and the force is changed to 50°, what magnitude force is necessary to tighten the bolt with the same torque? (assume both angles have 2 significant digits.)

AOR $\circ$------------------> $\cdots$-------   AOR $\circ$------------------> $\cdots$-------

$\vec{r}_1$ θ_1 $\vec{r}_2$ θ_2

$\vec{F}_1$ $\vec{F}_2$
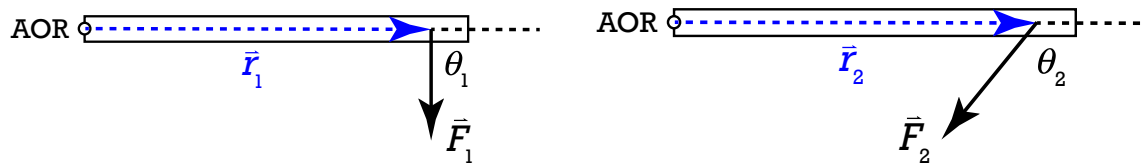
Knowns: $r_1 = 0.18m = r_2; F_1 = 29N; \theta_1 = 90°; \theta_2 = 180° - 50° = 130°; \tau_1 = \tau_2; F_2 = ?$

$$\vec{\tau} = \vec{r}\vec{F}\sin\theta \Rightarrow \tau = rF\sin\theta \left(magnitude\right)$$

$$\tau_1 = r_1F_1\sin\theta_1 = \left(0.18\right)\left(29\right)\sin 90 = 5.22N\cdot m \text{ (This is the torque applied by the original force.)}$$

$$\tau_1 = \tau_2 = r_2F_2\sin\theta_2 \Rightarrow F_2 = \frac{\tau_1}{r_2\sin\theta_2} = \frac{5.22}{\left(0.18\right)\sin\left(130\right)} = 37.8568 \approx \boxed{38N}$$

Part B) If a pipe is fitted to the wrench which increases the distance between the lag bolt and where the force is applied to 1.08 meters, what would the minimum magnitude force necessary be to cause the same torque as before?
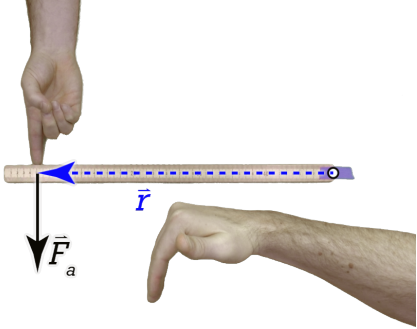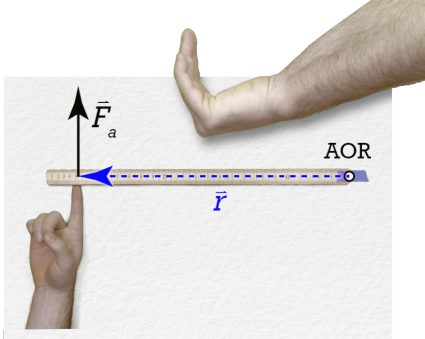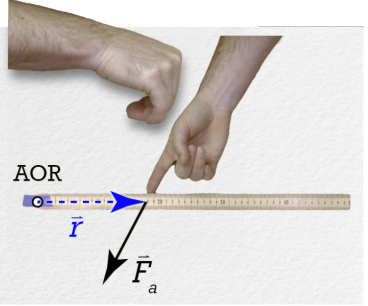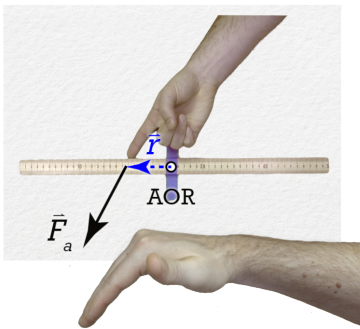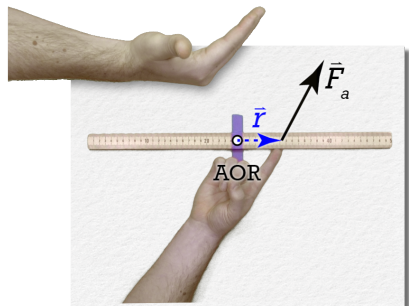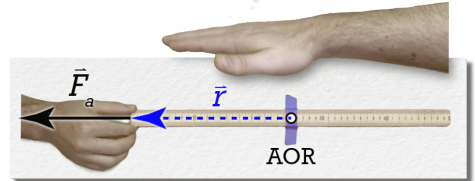

—————1.08 m—————
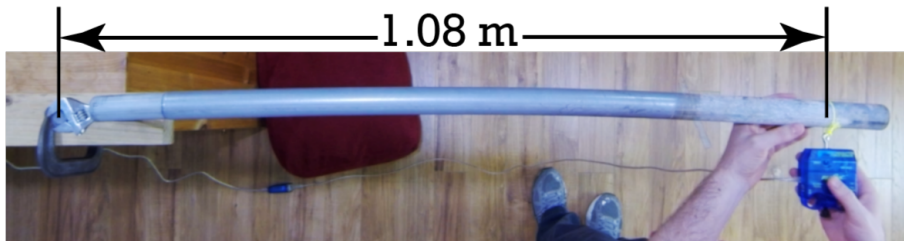
Minimum force means the angle needs to be 90° because $\sin 90 = 1$.
Any other force would result in a larger force necessary to produce the same torque.

$$\vec{\tau}_1 = \vec{\tau}_3 \Rightarrow \vec{F}_3 = \frac{\vec{\tau}_1}{r_3\sin\theta_3} = \frac{5.22}{\left(1.08\right)\sin 90} = 4.8\overline{3} \approx \boxed{4.8N}$$

The right hand rule for torque is used to find the direction of torque.
- Do not be too cool for the right hand rule. Limber up!!

1. Start with the fingers of your right hand at the axis of rotation.
2. Point your fingers toward the force.
3. Curl your fingers in the direction of the force.
4. Stick out your right thumb.
5. Your right thumb points in the direction of the torque.

Six demonstrations:

positive torque, out of the paper	negative torque, into the paper
negative torque, into the paper	positive torque, out of the paper
positive torque, out of the paper	torque is zero, so no torque direction. $$\vec{\tau} = \vec{r}\vec{F}\sin\theta = \left(0\right)\vec{F}\sin\theta = 0$$

Example: Kate, Geneve, and Ryan all push on a door as shown in the figure. Assuming the hinge is the axis of rotation of the door, determine the net torque caused by these three forces.

$$\vec{F}_K = 210N \qquad \vec{F}_R = 160N$$

45°   ←—0.74m—→   35°

AOR ←—0.37m—→

25°
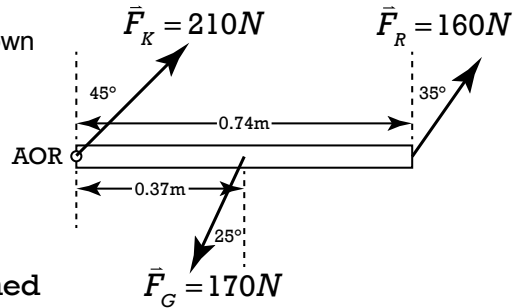
$$\vec{F}_G = 170N$$

$$\vec{\tau}_{net} = ?; \vec{F}_K = 210N; \vec{F}_G = 170N; \vec{F}_R = 160N;$$

$$\vec{\tau} = \vec{r}\vec{F}\sin\theta; \vec{r}_R = 0.74m; \vec{r}_G = 0.37m; \vec{r}_K = 0;$$

$$\theta_R = 90 - 35 = 55°; \theta_G = 90 + 25 = 115°; \theta_K = \text{undefined}$$

$$\vec{\tau}_K = \vec{r}_K \vec{F}_K \sin\theta_K = (0)(210)\sin\theta_K = 0$$

$$\vec{\tau}_G = \vec{r}_G \vec{F}_G \sin\theta_G = (0.37)(170)\sin(115) = 57.0068N \cdot m$$

$$\vec{\tau}_R = \vec{r}_R \vec{F}_R \sin\theta_R = (0.74)(160)\sin(55) = 96.9876N \cdot m$$

$$\vec{\tau}_{net} = ? = \vec{\tau}_K - \vec{\tau}_G + \vec{\tau}_R = 0 - 57.0068 + 96.9876 = 39.9808 \approx \boxed{4.0 \times 10^1 N \cdot m}$$

According to the Right Hand Rule, Geneve's torque is negative and Ryan's torque is positive.

Flipping Physics Lecture Notes:

Rotational Form of Newton's Second Law - Introduction

First we need to review Newton's Second Law of Motion: $\sum \vec{F} = m\vec{a}$

- Force and acceleration are both vectors.
- Includes not just force but the **net** force, meaning the addition of all the forces acting on an object.
- When you use this equation you have to identify:
  o what object(s) you are summing the forces on.
  o the direction in which you are summing the forces.

This is the rotational form of Newton's Second Law of Motion: $\boxed{\sum \vec{\tau} = I\vec{\alpha}}$

- Note the similarities between the original second law and the rotational form.
  o Torque is the rotational form of force.
  o Moment of Inertia or Rotational Inertia is the rotational form of inertial mass.
  o Angular acceleration is the rotational form of linear acceleration.
- Torque and angular acceleration are both vectors.
- Includes not just torque but the **net** torque, meaning the addition of all the torques acting on an object.
- When you use this equation you have to identify:
  o what object(s) you are summing the torques on.
  o the axis of rotation.
  o the positive torque direction.

Remember: Torque is the ability of a force to cause an angular acceleration of an object. Notice how the rotational form of Newton's Second Law of Motion shows exactly that. A net torque causes an angular acceleration. If you increase the net torque acting on an object without adjusting the moment of inertia or rotational inertia of the object, the angular acceleration of the object with increase.

If you want to hear the equation in words, you can say the angular acceleration of an object produced by a net torque is directly proportional to the magnitude of the net torque, in the same direction as the net torque, and inversely proportional to the moment of inertia or rotational inertia of the object.

$$\sum \vec{F} = m\vec{a} \Rightarrow \vec{a} = \frac{\sum \vec{F}}{m} \ \& \ \sum \vec{\tau} = I\vec{\alpha} \Rightarrow \vec{\alpha} = \frac{\sum \vec{\tau}}{I}$$

Have you ever struggled to describe Rotational Inertia to your students? Even worse, have you ever struggled to understand Rotational Inertia yourself. (I know I have. ☺) Did you know Rotational Inertia is the same as Moment of Inertia? Yeah, I'm with you there. I did not know the name had been changed until recently. However, I do think Rotational Inertia is a more logical phrase than Moment of Inertia. Well, if you would like some help with the concept of Rotational Inertia, then I highly suggest the Rotational Inertia Demonstrator from Arbor Scientific because it is an easy way to demonstrate the concept of rotational inertia. The demonstrator is composed of three pulleys of different sizes all centered around the same axle. Attached to the pulleys are four spokes on which four masses can be placed. The distance from the axle, or axis of rotation, of the four masses on the spokes can be adjusted.
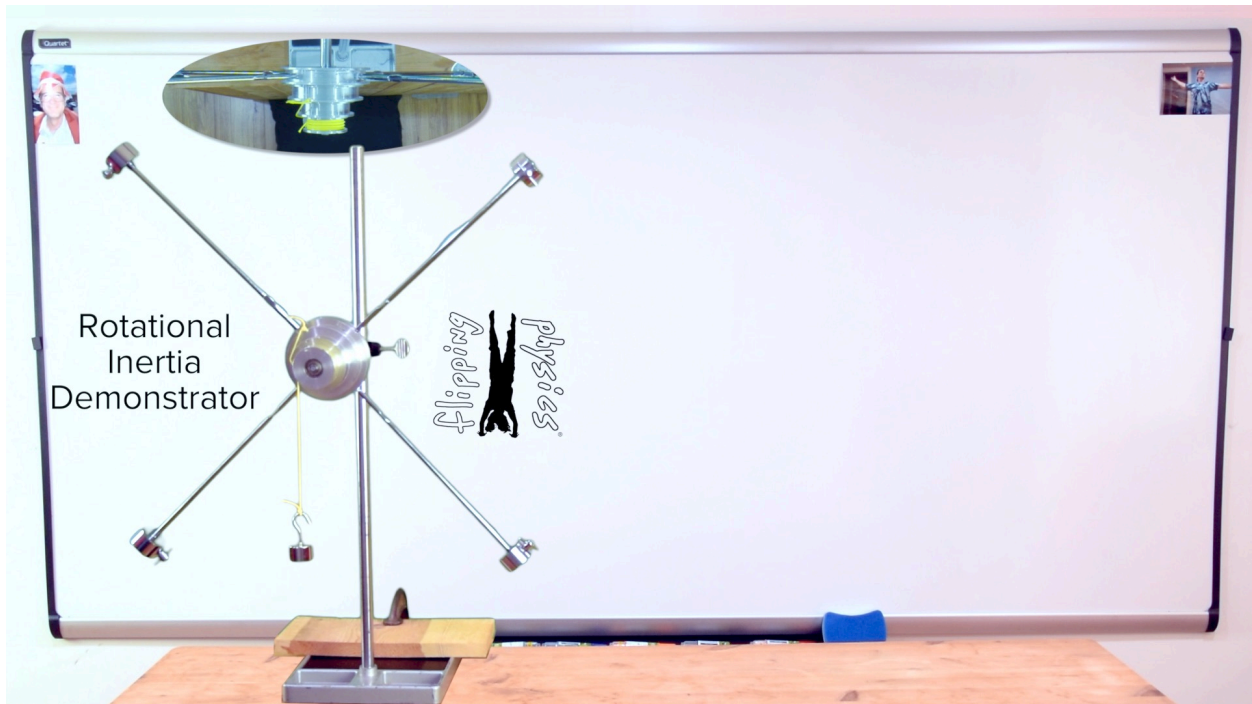
In order to understand rotational inertia, we should first review the equation for rotational inertia of a system of particles:

$$I_{particles} = \sum_i m_i r_i^2$$

The rotational inertia of a system of particles equals the sum of the quantity of the mass of each particle times the square of the distance each particle is from the axis of rotation. While the Rotational Inertia Demonstrator does not appear to be a system of particles, the equation for the rotational inertia of a system of particles helps us to understand how the rotational inertia of the demonstrator changes when we adjust the locations of the four adjustable masses. The closer the four adjustable masses are to the axle, or axis of rotation, the smaller the "r" value in the rotational inertia equation and the smaller the rotational inertia of the demonstrator.

We also need to review the Rotational Form of Newton's Second Law of Motion to better understand rotational inertia. The net torque acting on an object equals the rotational inertia of the object times the angular acceleration of the object. Please remember torque and angular acceleration are vectors.

$$\sum \vec{\tau} = I\vec{\alpha} \text{ (rotational)}$$

Notice the similarities to the Translational Form of Newton's Second Law of Motion. The net force acting on an object equals the inertial mass of the object times the linear acceleration of the object. Again, remember force and linear acceleration are vectors.

$$\sum \vec{F} = m\vec{a} \text{ (translational)}$$

Force is the ability to cause a linear acceleration of an object.
Torque is the ability of a force to cause an angular acceleration of an object.

Torque is the rotational equivalent of force.
Rotational inertia is the rotational equivalent of inertial mass.
Angular acceleration is the rotational equivalent of linear acceleration.

But, what does it mean that rotational inertia is the rotational equivalent of inertial mass? *Inertial mass* is the measurement of the resistance of an object to *linear* acceleration. Therefore, *rotational inertia* is the measurement of the resistance of an object to *angular* acceleration. In other words, the greater the rotational inertia of an object, the more that object will resist an angular acceleration. Referring back to the rotational inertia demonstrator, the farther the four adjustable masses are from the axis of rotation, the larger the "r" value in the equation for the rotational inertia of a system of particles, therefore the larger the rotational inertia of the demonstrator. The larger the rotational inertia of the demonstrator, the larger the resistance of the demonstrator to angular acceleration. In summary, the larger the distance the four adjustable masses are from the axle, the larger the rotational inertia, and therefore the larger the resistance of the demonstrator to angular acceleration.

This is demonstrated below by hanging a 100-gram mass from the largest pulley in two simultaneous demonstrations. In the demonstration on the left, the four adjustable masses are close to the axis of rotation and therefore the rotational inertia of the system is smaller. In the demonstration on the right, the four adjustable masses are farther from the axis of rotation and therefore the rotational inertia of the system is larger. When both demonstrators are simultaneously released from rest, because the net torque caused by the 100-gram masses is approximately the same, the demonstrator with the larger rotational inertia on the right has a smaller angular acceleration. In other words, the demonstrator with the larger rotational inertia speeds up rotationally at a slower rate. Going back to the Rotational Form of Newton's Second Law of Motion, because the net torque is almost the same, a larger rotational inertia results in a smaller angular acceleration: $\sum \vec{\tau} = I\vec{\alpha}$

$$\sum \vec{\tau} = I\vec{\alpha}$$

Larger I
Same $\sum \vec{\tau}$
Smaller $\vec{\alpha}$
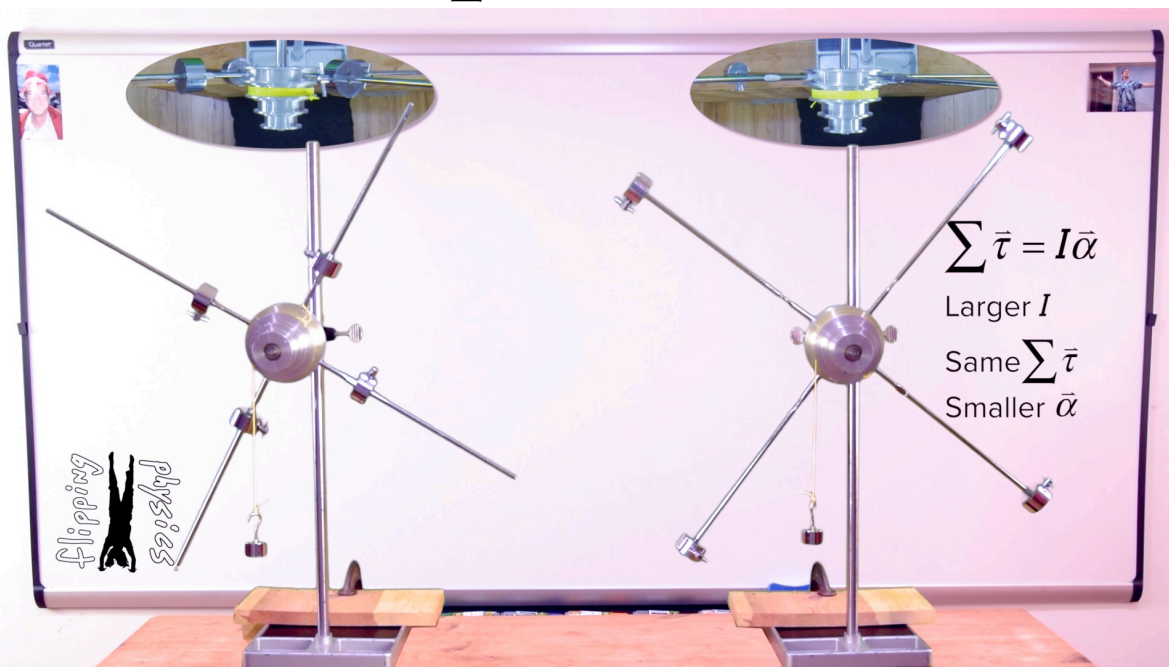
Notice we are always keeping the four adjustable masses the same distance from the axle, or axis of rotation. This is to keep the center of mass of the system at the axis of rotation of the system. When the four masses are not equally spaced from the axis of rotation, then the center of mass of the system is offset from the axis of rotation and the force of gravity acting on the system causes a torque on the system. The force of gravity causing a torque on the system makes understanding the demonstration much more complicated. In the examples shown below, the demonstrator on the left with four masses equally spaced from the axle rotates at almost a constant angular velocity. The demonstrator on the right has one mass farther from the axis of rotation and therefore the whole system actually becomes a physical pendulum. The system oscillates back and forth in simple harmonic motion. While this is interesting, it does not provide an obvious way to learn about rotational inertia. In summary, it is much easier to learn about rotational inertia from the demonstrator if all four masses are equally spaced from the axis of rotation.
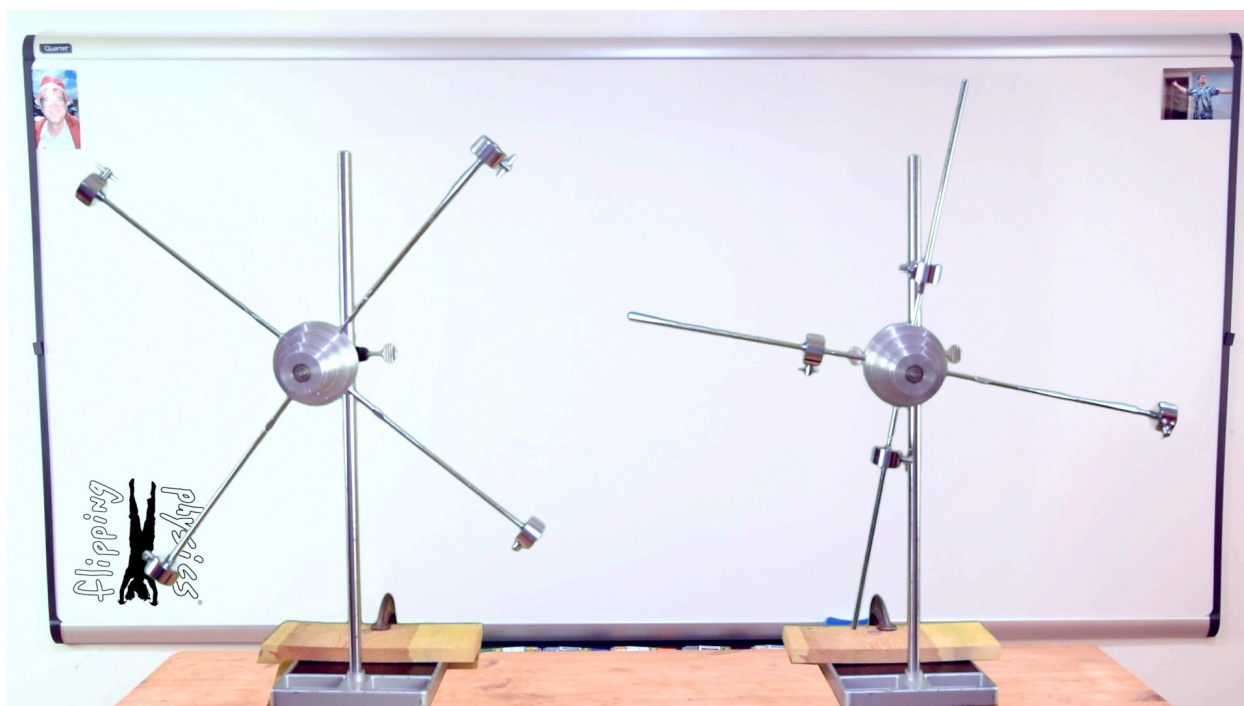

Let's look at another set of demonstrations below to learn about rotational inertia. As in the previous demonstration, on the right we have a 100-gram mass hanging from the largest pulley and all four adjustable masses far from the axis of rotation. On the left, all four adjustable masses are still far from the axis of rotation, however, the 100-gram mass is hanging from the smallest pulley instead. In other words, both rotational inertia demonstrators have the same rotational inertia and the force of gravity acting on the string is the same, however, the *net torque* acting on each demonstrator is different. Recall torque equals the "r" vector times the force causing the torque times the angle between the direction of the "r" vector and the direction of the force. The magnitude of the "r" vector is the distance from the axis of rotation to where the force is applied to the object:

$$\tau = rF\sin\theta$$

Because the 100-gram mass is hanging from the small pulley on the left and the large pulley on the right, the "r" vector for the small pulley is smaller and therefore the net torque acting on the demonstrator through the small pulley is less. Therefore, according to the Rotational Form of Newton's Second Law of Motion, the angular acceleration of the demonstrator on the left is less than the angular acceleration of the demonstrator on the right.

$$\sum \vec{\tau} = I\vec{\alpha}$$

same I
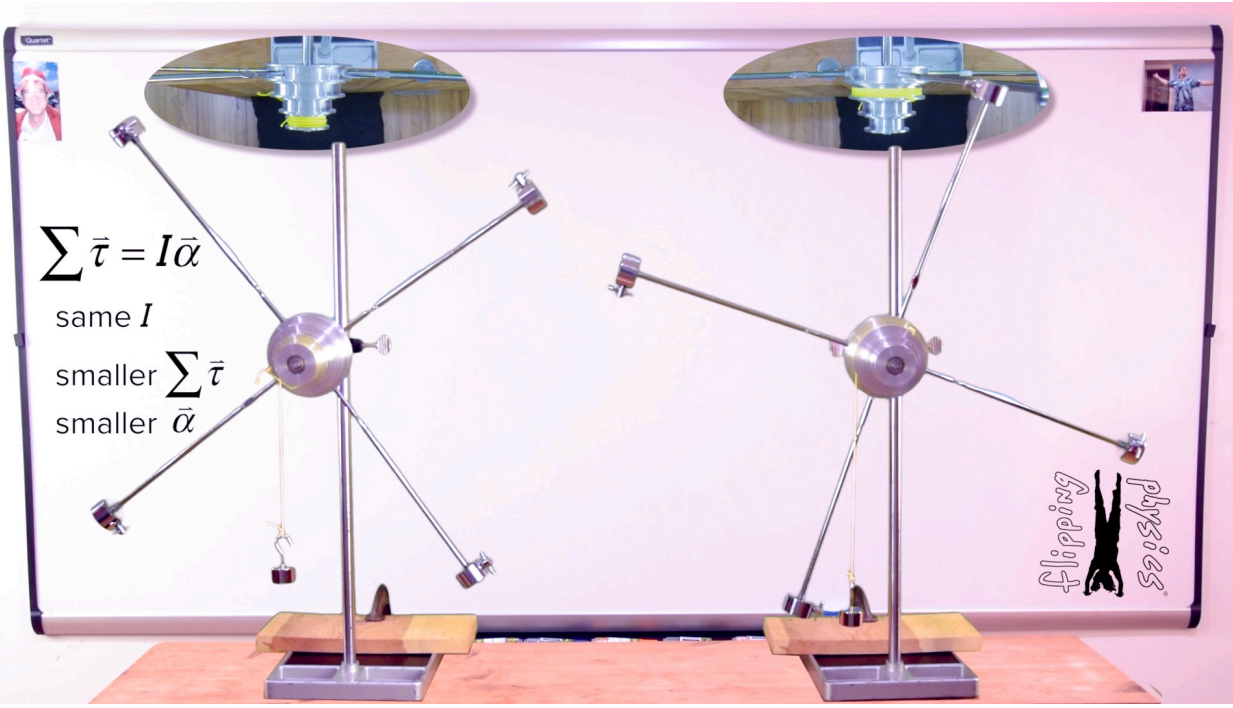
smaller $\sum \vec{\tau}$

smaller $\vec{\alpha}$

Our last set of demonstrations has both demonstrators with identical rotational inertias and masses hanging from the smallest pulleys. Also, both demonstrators have a 100-gram mass hanging over the left side of the pulley. However, the demonstrator on the right has a second mass, a 200-gram mass, hanging over the right side of the pulley. This means the demonstrator on the right has two different masses hanging off of the smallest pulley.

smallest → pulley

smallest → pulley

100 g →

100 g →

200 g ←
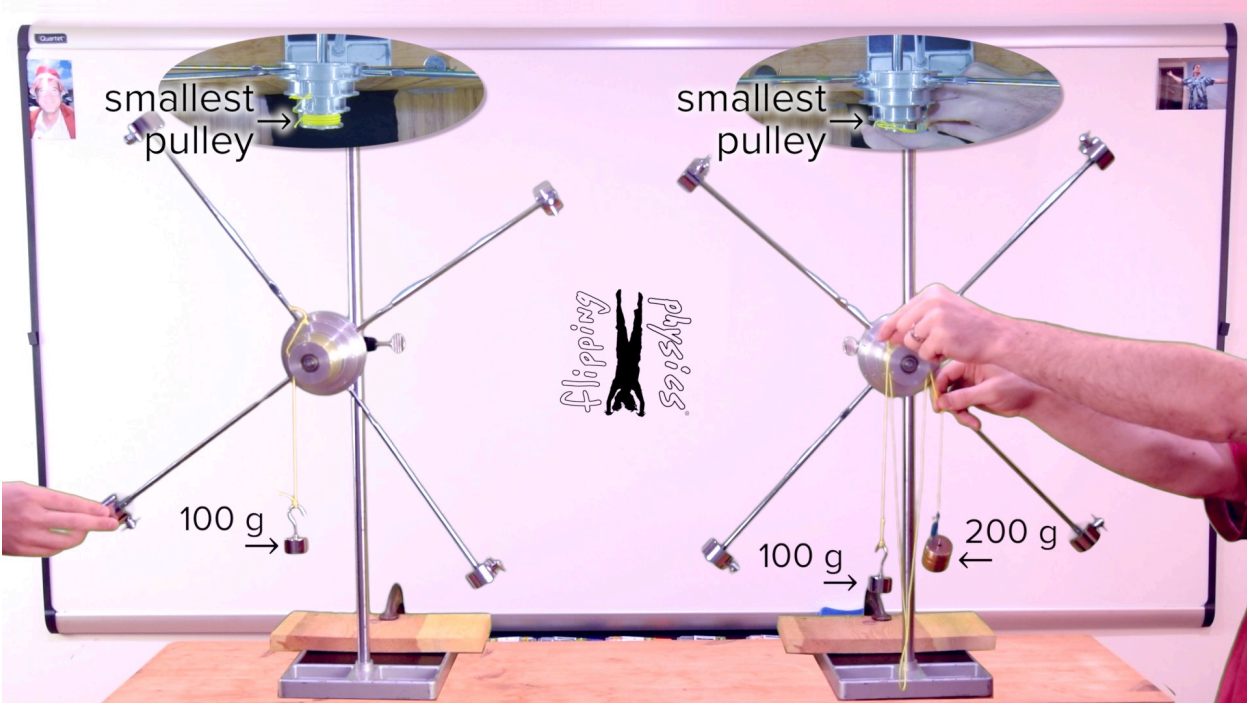
In order to determine what is going to happen, remember the Rotational Form of Newton's Second Law of Motion includes *net* torque not just torque. $\sum \vec{\tau} = I\vec{\alpha}$ In this example, the net torque from the two masses on the demonstrator on the right actually has roughly the same *magnitude* as the net torque acting on the demonstrator on the left, however, the directions are opposite from one another.

Again, both demonstrators have the same rotational inertia, are using the same pulley, and have a 100-gram mass hanging over the left side of the pulley. The pulley on the right adds a 200-gram mass hanging over the right side of the pulley. For the demonstrator on the right, the 100-gram mass hanging over the left side of the pulley essentially cancels out 100-grams of the 200-gram mass hanging over the right side of the pulley. This effectively means the right demonstrator essentially has a 100-gram mass hanging over the right side of the pulley. Therefore, the net torques on both demonstrators have essential the same magnitude and opposite directions. Therefore, the angular accelerations of both demonstrators should have roughly the same magnitude and opposite directions. You can see that is true in the demonstration.
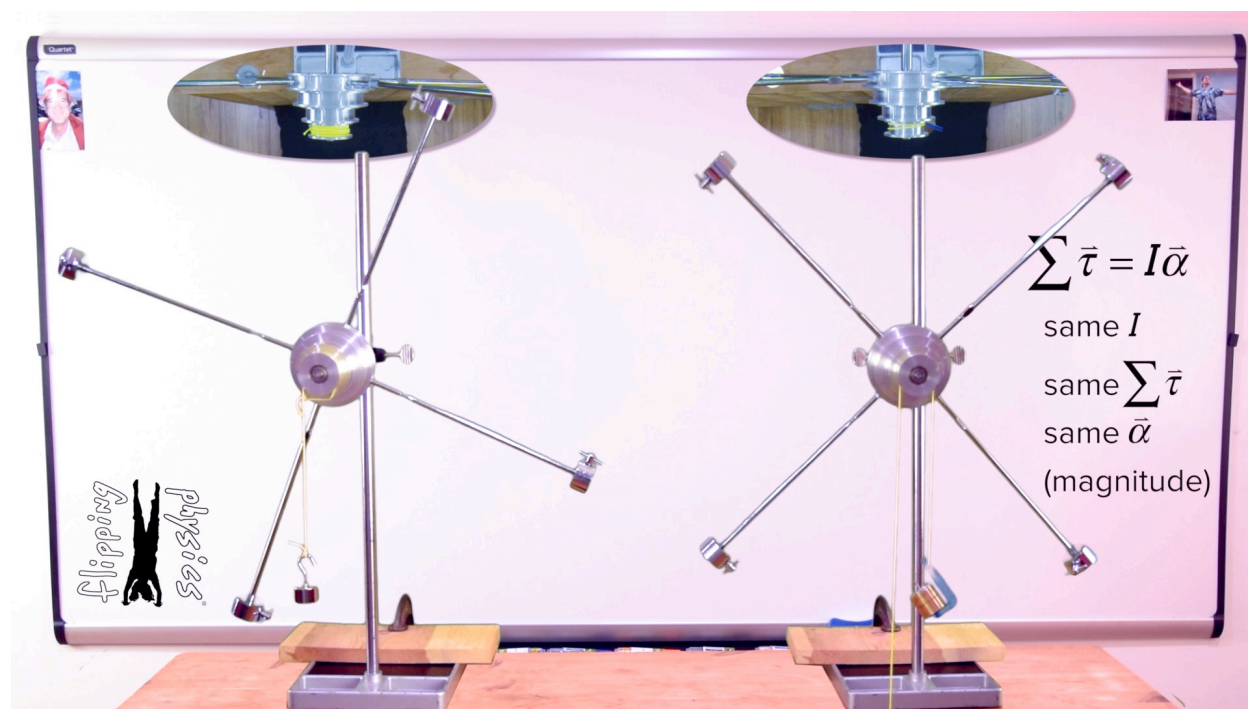
But why do the two demonstrators have "roughly" the same magnitude angular accelerations? Adding the 200-gram mass to the demonstrator on the right increases the total mass of the system. Because inertial mass is resistance to acceleration, increasing the total mass of the system actually decreases the angular acceleration of the system a little bit, even though the net torque should be roughly the same. Proving this requires drawing free body diagrams, summing the torques on the wheel, and summing the forces on each mass hanging, so I am not going to walk all the way thought that solution here.

There are many more ways you can make adjustments to the rotational inertia demonstrator to better help understand rotational inertia. For example, ask yourself what would happen to the angular acceleration of the demonstrator if the only change we make to it is to increase the mass hanging from the demonstrator? Increasing the mass hanging from the demonstrator increases the net torque acting on the demonstrator. The rotational inertia remains the same. Therefore, according to the Rotational Form of Newton's Second Law of Motion, $\sum \vec{\tau} = I\vec{\alpha}$, the angular acceleration of the demonstrator will increase.

What if the only change we make is to change the locations of the four adjustable masses from all being at their farthest extreme positions to having two of the adjustable masses near the axis of rotation and two adjustable masses far from the axis of rotation? Bringing two adjustable masses near the axis of rotation decreases the rotational inertia of the system and therefore, according to the Rotational Form of Newton's Second Law of Motion, the angular acceleration of the demonstrator will increase. Notice, this will only work when the two close adjustable masses are opposite one another and the two far adjustable masses are also opposite one another. If this is not the case, the center of mass of the rotational inertia demonstrator will not be at the axle, or axis of rotation, which is a problem we addressed earlier.

The pulley sizes of the rotational inertia demonstrator are provided by Arbor Scientific. They are 20.22 mm for the small pulley, 28.65 mm for the medium pulley, and 38.52 mm for the large pulley. Given this information, we can even predict which way the rotational inertia demonstrator will rotate if we were to hang 100-grams over one side of the large pulley and 200 grams over the other side of the small pulley. Before releasing the demonstrator, the angular acceleration of the demonstrator is zero because it is at rest. Therefore the torque caused by the 100-gram mass will be 0.3852 meters times 0.100 kilograms times 9.81 m/s^2 times the sine of 90 degrees which equals roughly 0.38 N.

$$r = 38.52mm \times \frac{1m}{1000mm} = 0.3852m \; \& \; m = 100g \times \frac{1kg}{1000g} = 0.1kg$$

$$\tau_{100-g} = rF_g \sin\theta = r(mg)\sin\theta = (0.3852)(0.1)(9.81)\sin(90) = 0.3778812 \approx 0.38N$$

The torque caused by the 200-gramm mass will be 0.2022 meters times 0.200 kilograms times 9.81 m/s^2 times the sine of 90 degrees which equals roughly 0.40 N.

$$r = 20.22mm \times \frac{1m}{1000mm} = 0.2022m \; \& \; m = 200g \times \frac{1kg}{1000g} = 0.2kg$$

$$\tau_{200-g} = rF_g \sin\theta = r(mg)\sin\theta = (0.2022)(0.2)(9.81)\sin(90) = 0.3967164 \approx 0.40N$$

Therefore, the net torque caused by both masses acting on the demonstrator *before it starts to accelerate* is the difference between these two torques because they act in opposite directions.

$$\sum \bar{\tau} = \tau_{200-g} - \tau_{100-g} = 0.3967164 - 0.3778812 = 0.0188352 \approx 0.02N$$

Therefore, because the torque caused by the 200-gram mass is larger than the torque caused by the 100-gramm mass, the rotational inertia demonstrator will rotate in the direction caused by the torque of the 200-gram mass.

Please realize these torque calculations are only correct while the demonstrator is at rest. Once the demonstrator begins to accelerate, the force of gravity and the force of tension acting on the mass hanging are no longer the same and we would need to draw free body diagrams and sum the forces on each hanging mass.

Thanks for reading and I hope you use the Rotational Inertia Demonstrator from Arbor Scientific to better understand rotational inertia!

Example: A uniform, solid disk that rotates about a frictionless axle at its center of mass is mounted on a wall so the plane of the disk is parallel to the wall. A string of negligible mass wraps around the disk and is pulled by a force of 11 N. If the radius of the disk is 0.18 m and the mass of the disk is 1.5 kg, what is the angular acceleration of the disk? The rotational inertia of a solid disk about its center of mass equals ½MR².

Knowns: $F_a = 11N; R_{disk} = 0.18m; M_{disk} = 1.5kg; \alpha = ?$

$$I_{disk} = \frac{1}{2}MR^2 = \frac{1}{2}(1.5)(0.18)^2 = 0.0243 kg \cdot m^2$$

Draw the Free Body Diagram and define the positive torque direction. →
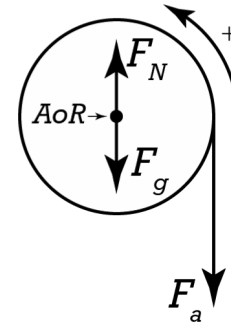
Sum the torques acting on the disk about its axis of rotation.

$$\sum \vec{\tau} = \cancel{\vec{\tau}_{F_N}} + \cancel{\vec{\tau}_{F_g}} - \vec{\tau}_{F_a} = I\vec{\alpha} \Rightarrow -\vec{\tau}_{F_a} = I\vec{\alpha}$$

Note: Because both F$_N$ and F$_g$ act on the axis of rotation, they both have an r value of zero and therefore produce no torque about the axle.

$$\Rightarrow -\vec{r}_{F_a}\vec{F}_a \sin\theta = -RF_a \sin(90) = I\alpha$$

$$\Rightarrow \alpha = -\frac{RF_a}{I} = -\frac{(0.18)(11)}{0.0243} = -81.4815 \approx \boxed{-81\frac{rad}{s^2}}$$

We are going to measure the rotational inertia (or moment of inertia) of a bike wheel. In order to do this we are going to attach a known mass to a string, wrap the string around the bike wheel, and let the mass apply a torque the bike wheel to angularly accelerate the wheel. We need to assume the axle of the bike wheel is frictionless.

FBD: F_N on bike wheel; F_g on bike wheel; F_g on hanging mass; F_T on bike wheel; F_T on hanging mass. r for both F_g on bike wheel and F_N on bike wheel are zero, so the torque caused by both of those forces is zero. Therefore, the only torque acting on the bike wheel is caused by the F_T of the hanging mass.

Sum the torques on only the wheel with the positive torque direction shown in the free body diagram:

$$\sum \tau_{\substack{wheel \\ AoR @ axle}} = +\tau_{F_T} = I\alpha \Rightarrow rF_T \sin\theta = RF_T \sin(90) = I\alpha \Rightarrow I = \frac{RF_T}{\alpha}$$

Sum the forces in the y-direction on just the hanging mass. Notice because of the way we have define positive torque that the positive y-direction is now down.

$$\sum F_{\substack{y \\ mass\,hanging}} = F_g - F_T = ma_y \Rightarrow mg - F_T = ma_y \Rightarrow F_T = mg - ma_y = m(g - a_y)$$

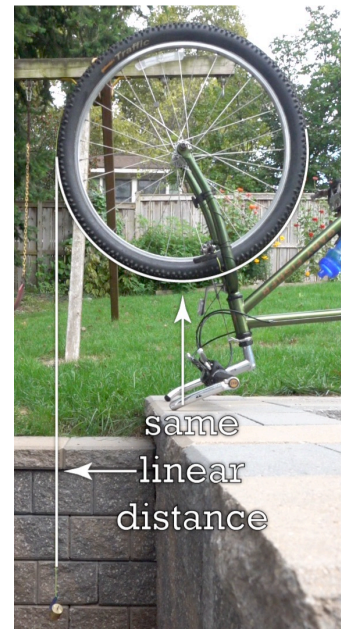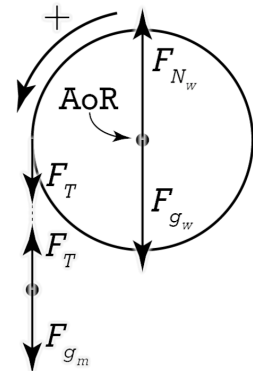Which we can substitute back in to the rotational inertia equation:

$$\Rightarrow I = \frac{RF_T}{\alpha} = \frac{Rm(g - a_y)}{\alpha}$$

Because the string is connected to the hanging mass and the outside edge of the wheel, as the hanging mass goes down, the edge of the wheel travels the same linear distance or arc length. This means the acceleration in the y-direction of the hanging mass is the same as the tangential acceleration of the rim of the wheel, therefore:

$$a_y = a_t = r\alpha = R\alpha \Rightarrow I = \frac{Rm(g - R\alpha)}{\alpha}$$

Everything in the equation for the net force on the hanging mass is constant. This means the force of tension acting on the wheel is constant. In addition, the "r" vector of the hanging mass and the angle in the torque equation are all constant. This means the net torque acting on the wheel is constant. The rotational inertia of the wheel is also constant. Therefore, according to the rotational form of Newton's second law of motion, the angular acceleration of the wheel is constant and we can use the uniformly angularly accelerated motion equations.

$$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha\Delta t^2 = (0)\Delta t + \frac{1}{2}\alpha\Delta t^2 \Rightarrow 2\Delta\theta = \alpha\Delta t^2 \Rightarrow \alpha = \frac{2\Delta\theta}{\Delta t^2}$$

Therefore, to solve for the moment of inertia of the bike wheel, we need to know:
- Wheel radius
- Hanging mass
- Acceleration due to gravity of the planet we are on
- Change in time while the hanging mass accelerates downward
- Change in angular position of the bike wheel associated with the same change in time

Knowns: $r_{wheel} = 0.332m; \; m_{hanging} = 0.205kg; \; g_{Earth} = 9.81\frac{m^2}{s}; \; \Delta t = 1.28\sec;$

$$\Delta \theta = 185° \times \frac{2\pi \, rad}{360°} = 3.228859 \, rad \; \& \; \alpha = \frac{2\Delta\theta}{\Delta t^2} = \frac{(2)(3.228859)}{1.28^2} = 3.941478\frac{rad}{s^2}$$
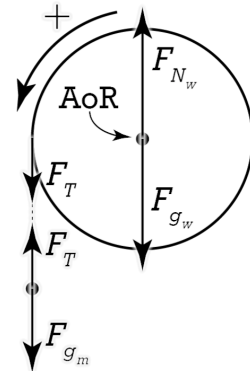
$$I = \frac{Rm(g - R\alpha)}{\alpha} = \frac{(0.332)(0.205)(9.81 - (0.332)(3.941478))}{3.941478} = 0.146800 \approx \boxed{0.147 kg \cdot m^2}$$

This is a continuation of "(1 of 2) Measuring the Rotational Inertia of a Bike Wheel." The following is a very rough summary of what was done in that lecture. Please watch that lecture before embarking on this video's learning adventure. https://www.flippingphysics.com/rotational-inertia-bike-wheel-1.html

Knowns: $\alpha = 3.941478 \dfrac{rad}{s^2}$; $r_{wheel} = 0.332m$; $m_{hanging} = 0.205kg$

$$\sum \tau_{\substack{on\ wheel \\ AoR@axle}} = +\tau_{F_T} = I\alpha \Rightarrow I = \frac{RF_T}{\alpha}$$

$$\sum F_{\substack{y \\ mass\ hanging}} = F_g - F_T = ma_y \Rightarrow F_T = m\left(g - a_y\right)$$

$$\Rightarrow I = \frac{Rm\left(g - R\alpha\right)}{\alpha} = 0.146800 \approx \boxed{0.147kg \cdot m^2}$$

Does this answer make sense? Recall that, about their center of masses and long cylindrical axes, the rotational inertias of spheres and cylinders were always a fraction times the mass of the object times the radius of the object squared. For example, about their long, cylindrical axes:

$$I_{thin\ hoop} = MR^2 \ \& \ I_{solid\ disk} = \frac{1}{2}MR^2$$

I would estimate that this bike tire is roughly halfway between a solid disc and a thin hoop. We could estimate this rotational inertia as

$I_{bike\ wheel} \approx \dfrac{3}{4}MR^2$. Let's test that:

$$m_{wheel} = 1.96kg; R_{wheel} = 0.332m; I_{wheel} = XM_{wheel}R_{wheel}^2 \Rightarrow X = \frac{I_{wheel}}{M_{wheel}R_{wheel}^2}$$

$$\Rightarrow X = \frac{0.146800}{\left(1.96\right)\left(0.332\right)^2} = 0.679505 \Rightarrow I_{bike\ wheel} \approx 0.680MR^2 \approx \frac{3}{4}MR^2$$

Notice it would be **incorrect** to sum the torques on the whole system at once (wheel and hanging mass):

$$\sum \tau_{\substack{on\ all \\ AoR@axle}} = +\bar{\tau}_{F_{T_{wheel}}} \cancel{\ } - \bar{\tau}_{F_{T_{mass\ hanging}}} \cancel{\ } + \bar{\tau}_{F_{g_{mass\ hanging}}} = +\bar{\tau}_{F_{g_{mass\ hanging}}} = I\bar{\alpha}$$

This is **incorrect** because the hanging mass does not have a rotational inertia about the axis of rotation of the bicycle wheel.

Part B) Determine the force of tension in the string while the wheel is angularly accelerating.

There are two equations we could use to solve for this now:

$$I = \frac{RF_T}{\alpha} \Rightarrow F_T = \frac{I\alpha}{R} = \frac{(0.146800)(3.941478)}{0.332} = 1.742793 \approx \boxed{1.74N}$$

$$F_T = m(g - R\alpha) = (0.205)(9.81 - (0.332)(3.941478)) = 1.742793 \approx 1.74N$$

Notice what happens *before* the wheel is allowed to angularly accelerate, in other words, while the wheel is at rest. Let's sum the forces on the hanging mass.

$$\sum F_y = F_g - F_T = ma_y = m(0) = 0 \Rightarrow F_T = F_g = mg = (0.205)(9.81) = 2.01105 \approx 2.01N$$

The force of tension equals the force of gravity, 2.01N, before the wheel begins to angularly accelerate and then decreases to 1.74 N while the wheel is angularly accelerating.

	Predicted	Measured
at rest	$F_T \approx 2.01N$	$F_T \approx 1.9N$
accelerating	$F_T \approx 1.74N$	$F_T \approx 1.6N$
difference	$\Delta F_T \approx 0.3N$	$\Delta F_T \approx 0.3N$

Translational Equilibrium:

- $\sum \vec{F} = 0 = m\vec{a} \Rightarrow \vec{a} = 0$
  - The object is either
    - **at rest** or
    - moving at a **constant velocity**.
      - Note: The mass of an object cannot be zero.
  - Remember to identify
    - **object(s)** you are summing the forces on and
    - the **direction** you are summing the forces in.

Rotational Equilibrium:

- $\sum \vec{\tau} = 0 = I\vec{\alpha} \Rightarrow \vec{\alpha} = 0$
  - The object is either
    - **at rest (not rotating)** or
    - moving at a **constant *angular* velocity**.
      - Note: The rotational inertia of an object cannot be zero.
  - Remember to identify
    - **object(s)** you are summing the torques on,
    - the **axis of rotation,** and
    - the **direction** you are summing the torques in.
  - Realize the rotational inertia of an object cannot be zero.

Static Equilibrium:

- The object is at rest (and therefore not rotating).
- The object is in both translational and rotational equilibrium.
- The net torque equals zero about **any axis of rotation**.

Translational Equilibrium:

$$\sum \vec{F} = 0 = m\vec{a} \Rightarrow \vec{a} = 0$$

- At Rest or
- Constant Velocity

Identify: Object and Direction

Rotational Equilibrium:

$$\sum \vec{\tau} = 0 = I\vec{\alpha} \Rightarrow \vec{\alpha} = 0$$

- At Rest (not rotating) or
- Constant *Angular* Velocity

Identify: Object, Direction, and Axis of Rotation

Static Equilibrium:

$$\sum \vec{F} = 0 = m\vec{a} \Rightarrow \vec{a} = 0 \ \& \ \sum \vec{\tau} = 0 = I\vec{\alpha} \Rightarrow \vec{\alpha} = 0$$

- At rest and not rotating
- $\sum \vec{\tau} = 0$ about ***any*** Axis of Rotation!

Example: A uniform 0.093 kg meterstick is supported at the 15 cm and 92 cm marks. When a 0.250 kg object is placed at the 6.0 cm mark, what are the magnitudes of the forces supporting the meterstick?

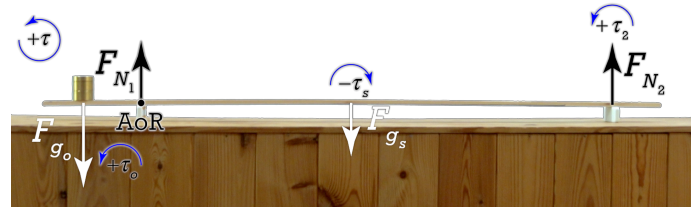$m_s = 0.093 kg; m_o = 0.250 kg; F_{N_1} = ?; F_{N_2} = ?$

The system is at rest, so it is in both translational and rotational equilibrium. Therefore, the net force equals zero and the net torque about any axis of rotation equals zero. This special case is called *static equilibrium.*

Sum the forces in the y-direction on the meterstick.

$$\sum F_y = -F_{g_o} + F_{N_1} - F_{g_s} + F_{N_2} = ma_y = m(0) = 0$$

$$\Rightarrow -m_o g + F_{N_1} - m_s g + F_{N_2} = 0$$

Both force normals are unknowns, so we need to put this equation in our equation holster and sum the torques on the meterstick about force normal #1. Note: Counterclockwise or out of the page is positive.

$$\sum \tau_{\substack{meterstick \\ AoR @ F_{N_1}}} = \tau_o + \tau_1 - \tau_s + \tau_2 = I\alpha = I(0) = 0$$

Torque directions:
- The Force of gravity of the object would cause the meterstick to rotate counterclockwise or out of the page, so the torque caused by force of gravity of the object is positive.
- Force normal #1 acts right at the axis of rotation, therefore, the "r" value for force normal #1 is zero, and the torque caused by force normal #1 is zero and has no direction.
- The force of gravity of the stick and force of gravity 2 would both cause the meterstick to rotate clockwise or into the page, so the torques caused by force of gravity of the stick and force of gravity #2 are both negative.

$$\Rightarrow r_o F_{g_o} \sin\theta_o - r_s F_{gs} \sin\theta_s + r_2 F_{N_2} \sin\theta_2 = 0 \,\&\, \theta_o = \theta_s = \theta_2 = 90° \,\&\, \sin(90°) = 1$$

$$\Rightarrow r_o m_o g - r_s m_s g + r_2 F_{N_2} = 0 \Rightarrow r_2 F_{N_2} = r_s m_s g - r_o m_o g \Rightarrow F_{N_2} = \frac{r_s m_s g - r_o m_o g}{r_2}$$

$$15 = 6 + r_o \Rightarrow r_o = 15 - 6 = 9cm \approx 0.09m \,\&\, 15 + r_s = 50 \Rightarrow r_s = 50 - 15 = 35cm$$

$$15 + r_2 = 92 \Rightarrow r_2 = 92 - 15 = 77cm$$

$$F_{N_2} = \frac{(35)(0.093)(9.81) - (9)(0.25)(9.81)}{77} = 0.12804 \approx \boxed{0.13N}$$

$$F_{N_2} = \frac{r_s m_s g - r_o m_o g}{r_2} \Rightarrow \frac{(cm)(kg)\left(\frac{m}{s^2}\right) - (cm)(kg)\left(\frac{m}{s^2}\right)}{cm} = (kg)\left(\frac{m}{s^2}\right) = N$$

And, going back to our equation holster:

$$-m_o g + F_{N_1} - m_s g + F_{N_2} = 0 \Rightarrow F_{N_1} = +m_o g - F_{N_2} + m_s g$$

$$\Rightarrow F_{N_1} = (0.25)(9.81) - 0.12804 + (0.093)(9.81) = 3.23679 \approx \boxed{3.2N}$$

Flipping Physics Lecture Notes:

Placing the Fulcrum on a Seesaw

Example: A 200.0 g mass is placed at the 20.0 cm mark on a uniform 93 g meterstick. A 100.0 g mass is placed at the 90.0 cm mark. Where on the meterstick should the fulcrum be placed to balance the system?

$$m_2 = 200.0g \ @ \ 20.0cm; \ m_s = 93g; \ m_1 = 100.0g \ @ \ 90.0cm; \ x = ?$$

The system is at rest, so it is in both translational and rotational equilibrium. Therefore, the net force equals zero and the net torque about any axis of rotation equals zero. This special case is called *static equilibrium*.

Sum the forces in the y-direction on the meterstick.

$$\sum F_y = F_N - F_{g_2} - F_{g_s} - F_{g_1} = ma_y = m(0) = 0$$

$$\Rightarrow F_N = F_{g_2} + F_{g_s} + F_{g_1} = m_2g + m_sg + m_1g$$
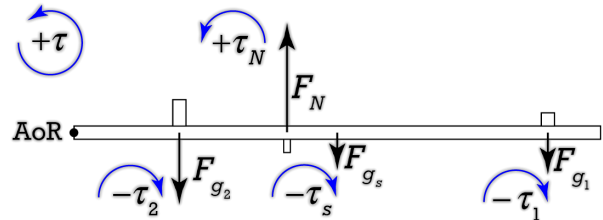
$$\Rightarrow F_N = g(m_2 + m_s + m_1)$$
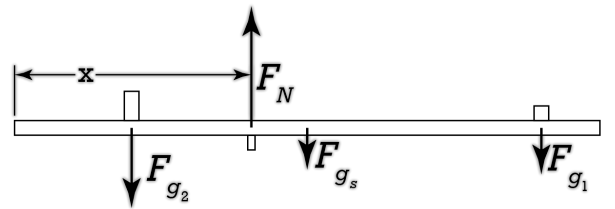
Now we sum the torques on the meterstick with the axis of rotation at the left end. Assume counterclockwise, or out of the page, is positive.

$$\sum \tau_{\substack{meterstick \\ AoR \ @ \ left \, end}} = -\tau_2 + \tau_N - \tau_s - \tau_1 = I\alpha = I(0) = 0$$

Torque directions:
- The force normal would cause the meterstick to rotate counterclockwise or out of the page, so the torque caused by force normal is positive.
- The force of gravity 2, force of gravity of the stick and force of gravity 1 would each cause the meterstick to rotate clockwise or into the page, so these three torques are each negative.

$$\Rightarrow -r_2 F_{g_2} \sin\theta_2 + r_N F_N \sin\theta_N - r_s F_{g_s} \sin\theta_s - r_1 F_{g_1} \sin\theta = 0$$

$$\Rightarrow -r_2 m_2 g \sin\theta_2 + r_N F_N \sin\theta_N - r_s m_s g \sin\theta_s - r_1 m_1 g \sin\theta = 0$$

$$\theta_2 = \theta_N = \theta_s = \theta_1 = 90° \ \& \sin(90°) = 1$$

$$\Rightarrow -r_2 m_2 g + r_N F_N - r_s m_s g - r_1 m_1 g = 0$$

$$\Rightarrow r_N F_N = r_2 m_2 g + r_s m_s g + r_1 m_1 g = g(r_2 m_2 + r_s m_s + r_1 m_1) \Rightarrow r_N = \frac{g(r_2 m_2 + r_s m_s + r_1 m_1)}{F_N}$$

$$\Rightarrow r_N = \frac{g(r_2 m_2 + r_s m_s + r_1 m_1)}{g(m_2 + m_s + m_1)} = \frac{r_2 m_2 + r_s m_s + r_1 m_1}{m_2 + m_s + m_1} = \frac{(20)(200) + (50)(93) + (90)(100)}{200 + 93 + 100}$$

$$\Rightarrow r_N = 44.9109 \approx \boxed{45cm}$$

Alternate solution without using Newton's Second Law:

Now we sum the torques on the meterstick with the axis of rotation at the Force Normal. Assume counterclockwise or out of the page is positive.
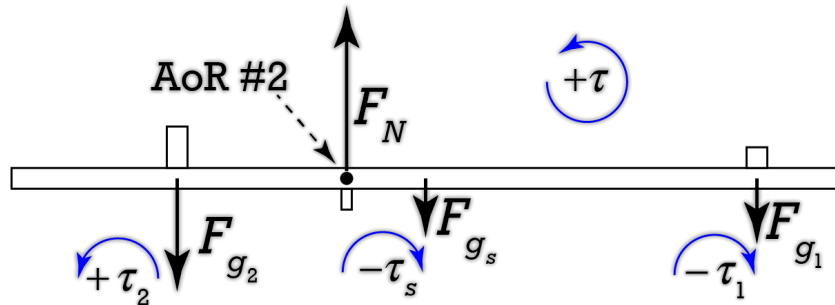
$$\sum \tau_{\substack{meterstick \\ AoR\,\#2\,@\,fulcrum}} = \tau_2 + \cancel{\tau_N} - \tau_s - \tau_1 = I\alpha = I(0) = 0$$

Torque directions:
- Force of gravity 2 would cause the meterstick to rotate counterclockwise or out of the page, so the torque caused by force of gravity 2 is positive.
- The force normal acts right at the axis of rotation, therefore, the "r" value for the force normal is zero, and the torque caused by the force normal is zero and has no direction.
- The force of gravity of the stick and force of gravity 2 would both cause the meterstick to rotate clockwise or into the page, so the torques caused by force of gravity of the stick and force of gravity #2 are both negative.
-

$$\Rightarrow r_2 F_{g_2} \sin\theta_2 - r_s F_{g_s} \sin\theta_s - r_1 F_{g_1} \sin\theta_1 = 0$$

$$\theta_2 = \theta_s = \theta_1 = 90° \,\&\, \sin(90°) = 1$$

$$\Rightarrow r_2 m_2 \cancel{g} - r_s m_s \cancel{g} - r_1 m_1 \cancel{g} = 0 \Rightarrow r_2 m_2 - r_s m_s - r_1 m_1 = 0$$

(everybody brought g, the acceleration to gravity, to the party!)

Define x as the distance from the left end of the meterstick to the axis of rotation.

$$x = 20 + r_2 \Rightarrow r_2 = x - 20 \,\&\, 50 = x + r_s \Rightarrow r_s = 50 - x \,\&\, 90 = x + r_1 \Rightarrow r_1 = 90 - x$$

$$r_2 m_2 - r_s m_s - r_1 m_1 = 0 \Rightarrow (x - 20)(200) - (50 - x)(93) - (90 - x)(100) = 0$$
$$\Rightarrow 200x - 4000 - 4650 + 93x - 9000 + 100x = 0$$
$$\Rightarrow 200x + 93x + 100x - 4000 - 4650 - 9000 = 0$$
$$\Rightarrow (200 + 93 + 100)x - (17650) = 0 \Rightarrow 393x = 17650$$

$$\Rightarrow x = \frac{17650 g \cdot cm}{393 g} = 44.911 cm \approx \boxed{45cm}$$

Example: What is the closest to the end of a 93 g uniform meterstick you can place a 200.0 g object and have the system stay balanced? The meterstick is supported at the 20.0 cm and 80.0 cm marks.

$m_s = 93g; m_o = 200.0g;$ supports at 20.0cm and 80.0 cm; $x = ?$

The system is at rest, so it is in both translational and rotational equilibrium. Therefore, the net force equals zero and the net torque about any axis of rotation equals zero. This special case is called *static equilibrium*. Let's sum the torques on the meterstick about the 20.0 cm mark which is location of one of the supports, Force Normal #1. Assume counterclockwise or out of the page is positive.
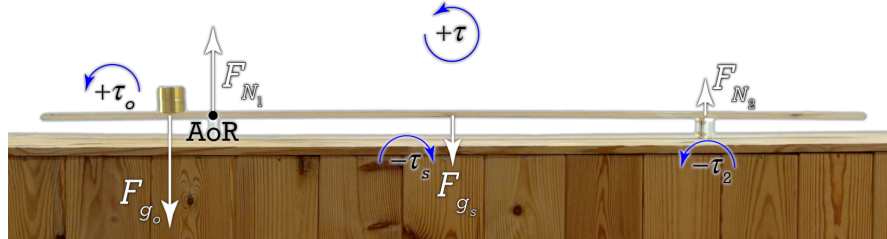
$$\sum \tau_{AoR @ \atop 20.0cm} = \tau_o + \tau_1 - \tau_s + \tau_2 = I\alpha = I(0) = 0 \Rightarrow \tau_o - \tau_s = 0 \Rightarrow \tau_o = \tau_s$$

Torque directions:
- The Force of gravity of the object would cause the meterstick to rotate counterclockwise or out of the page, so the torque caused by force of gravity of the object is positive.
- Force normal #1 acts right at the axis of rotation, therefore, the "r" value for force normal #1 is zero, and the torque caused by force normal #1 is zero and has no direction.
- The force of gravity of the stick and force of gravity 2 would both cause the meterstick to rotate clockwise or into the page, so the torques caused by force of gravity of the stick and force of gravity #2 are both negative.

As we move the object closer to the left end of the meterstick, the magnitude of force normal #2 decreases. When force normal #2 is reduced to zero, the object has reached it closest point to the left end of the meterstick, any farther left and the system would unbalance. Therefore, in this problem, force normal #2 is zero and the torque caused by force normal #2 is also zero.

$$\Rightarrow r_o F_{g_o} \sin\theta_o = r_s F_{g_s} \sin\theta_s$$

$$\Rightarrow r_o m_o g \sin(90) = r_s m_s g \sin(90) \& \theta_o = \theta_s = 90° \& \sin(90°) = 1$$

(everybody brought g, the acceleration to gravity, to the party!)

$$\Rightarrow r_o m_o = r_s m_s \Rightarrow r_o = \frac{r_s m_s}{m_o} = \frac{(30)(93)}{(200)} = 13.95cm$$

Note: $50 = 20 + r_s \Rightarrow r_s = 50 - 20 = 30cm$

$$20 = x + r_o \Rightarrow x = 20 - r_o = 20 - 13.95 = 6.05 \approx 6.0cm$$

Answer: 6.0cm from the end of the meterstick

Flipping Physics Lecture Notes:

Graphing the Rotational Inertia of an Irregular Shape

We have discussed the equations for the rotational inertia of common shapes. See: Moments of Inertia of Rigid Objects with Shape - https://www.flippingphysics.com/moment-of-inertia-rigid-objects.html However, we should also know how to measure the rotational inertia of irregular shapes. The Rotational Inertia Demonstrator from Arbor Scientific is a pulley; however, it does not fit any of the standard shapes. Our goal is to create a graph where the slope of the best-fit line is the rotational inertia of the rotational inertia demonstrator. Let's start with free body diagrams.

We can sum the torques on the pulley with an axis of rotation and the center of the pulley and define counterclockwise, or out of the board, as positive. Because they both act on the axis of rotation, the force normal and force of gravity which act on the pulley cause zero torque on the pulley. The torque caused by the force of tension on the pulley causes the pulley to rotate in the positive direction.
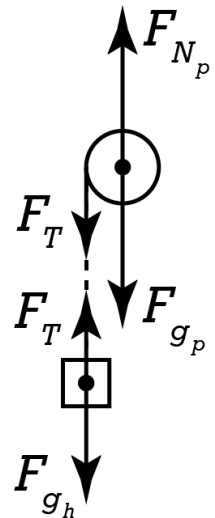
$$\sum \vec{\tau}_{pulley \atop AoR @ pulley\ center} = \vec{\tau}_{F_T} = I\vec{\alpha} \Rightarrow rF_T \sin\theta = I\alpha \Rightarrow RF_T \sin(90) = I\alpha \Rightarrow RF_T = I\alpha$$

Compare this equation to the slope intercept form of a line equation:

$$y = (slope)x + b \Rightarrow y = RF_T; \ slope = I; \ x = \alpha; \ b = 0$$

The pulley has three different radii: $R_1 = 0.0202m; \ R_2 = 0.0286m; \ R_3 = 0.0385m$

The force of tension can be measured using a force sensor as a part of the hanging mass. Because the string has the same force of tension on both ends, the force of tension we measure on the hanging mass is the same as the force of tension which acts on the pulley.

The angular acceleration of the pulley needs to be measured using a uniformly angularly accelerated motion equation:

$$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha\Delta t^2 = \frac{1}{2}\alpha\Delta t^2 \Rightarrow \alpha = \frac{2\Delta\theta}{\Delta t^2}$$

Therefore we need change in time, angular displacement, and zero initial angular velocity.

Here is a sample calculation for the angular acceleration of the first trial:

$$\Delta t = 55 frames \times \frac{1sec}{60 frames} = 0.91\overline{6}sec; \ \Delta\theta = 2rev \times \frac{2\pi rad}{1rev} = 4\pi rad; \ \omega_i = 0$$

$$\alpha = \frac{2\Delta\theta}{\Delta t^2} = \frac{(2)(4\pi)}{(0.91\overline{6})^2} = 29.910\frac{rad}{s^2}$$

Trial	Hanging Mass (kg)	Pulley Radius (m)	Force of Tension (N)	α (rad/s^2)	R * F$_T$ (m·N)
1	0.103	0.0202	0.955	29.9	0.0193
2	0.153	0.0202	1.365	44.7	0.0276
3	0.203	0.0202	1.762	56.5	0.0356
4	0.253	0.0202	2.131	66.1	0.0431
5	0.103	0.0286	0.904	37.7	0.0258
6	0.153	0.0286	1.268	52.5	0.0363
7	0.203	0.0286	1.566	69.8	0.0448
8	0.253	0.0286	1.871	78.3	0.0535
9	0.103	0.0385	0.829	46.7	0.0319
10	0.153	0.0385	1.122	66.1	0.0432
11	0.203	0.0385	1.361	76.0	0.0524
12	0.253	0.0385	1.566	88.4	0.0603
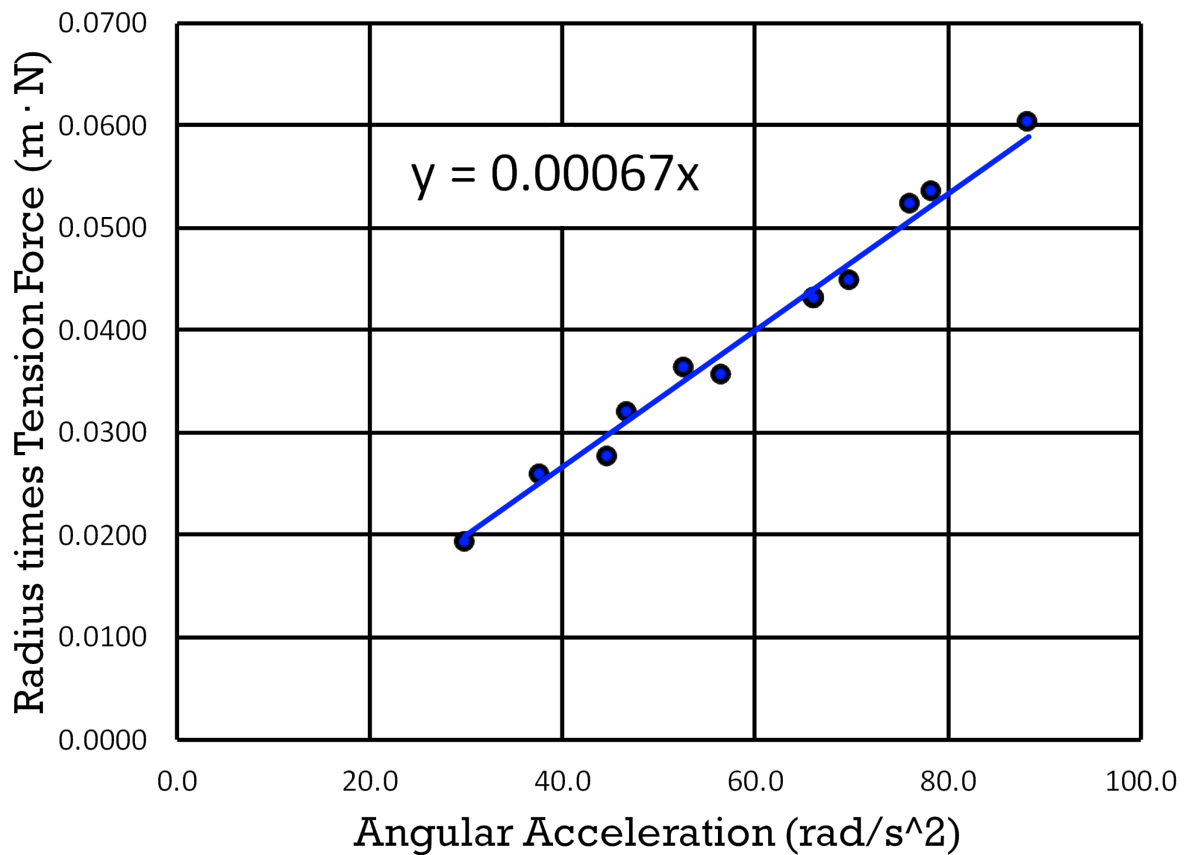
y = 0.00067x

The best fit line equation y = 0.00067x means $RF_T = 0.00067\alpha$ which means, because $RF_T = I\alpha$, the rotational inertia of the Rotational Inertia Demonstrator is 0.00067 kg·m^2.

Confirming the units:

$$RF_T = I\alpha \Rightarrow I = \frac{RF_T}{\alpha} \Rightarrow \frac{m \cdot N}{\frac{rad}{s^2}} = \left(m \cdot \frac{kg \cdot m}{s^2} \right)\left(\frac{s^2}{rad} \right) = kg \cdot m^2$$

Previously we determined the rotational inertia of the Rotational Inertia Demonstrator from Arbor Scientific. In order to do so, we measured the force of tension acting on the mass hanging which has the same magnitude as the force of tension acting on the pulley. Let's take a look at how the force of tension changes depending on the angular acceleration of the pulley. We need to start, of course, with free body diagrams. Remember we defined counterclockwise, or out of the board, as the positive torque direction.

Here is the graph for the measured force of tension as a function of time for the first trial:

$F_{avg} \approx 1.013N$

$F_{avg} \approx 0.955N$
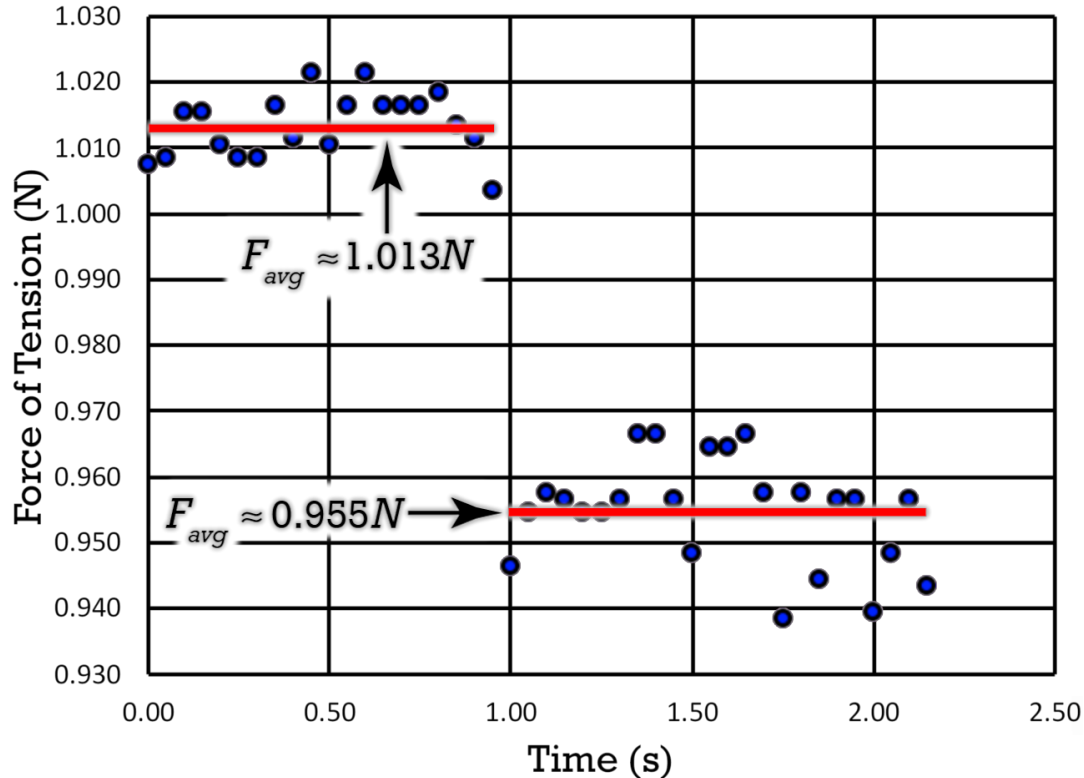
The hanging mass is released just before 1.00 seconds. The acceleration of the system before that is zero because the system is at rest. Therefore:

$$\sum F_{y\,mass} = F_g - F_T = ma_y = m(0) = 0 \Rightarrow F_T = F_g = mg = (0.103)(9.81) = 1.01043 \approx 1.01N$$

After the hanging mass is released, the force of tension changes because the acceleration of the system is no longer zero:

$$\sum F_{y\,mass} = F_g - F_T = ma_y \Rightarrow F_T = F_g - ma_y = mg - ma_y = m(g - a_y)$$

Be careful of direction here. Remember, we defined counterclockwise or out of the board as positive, therefore, the direction the hanging mass is moving is the positive direction. This is why the force of gravity and acceleration of the hanging mass are both positive when we sum the forces. The way we determine the linear acceleration of the hanging mass is by using the tangential acceleration equation.

$$a_y = a_t = r\alpha = R\alpha$$

$$\Rightarrow F_T = m(g - R\alpha) = (0.103)(9.81 - (0.0202)(29.910)) = 0.948199 \approx 0.948N$$

This predicted value compares quite well to our average force of tension measurement of 0.955 N.

The equation for rotational inertia of a rigid object with shape is: $I = \int r^2 \, dm$

We are going to determine the rotational inertia of a long, thin, uniform density rod. For that we need to use linear mass density, λ.

$$\lambda = \frac{M}{L} = \frac{dm}{dx} \Rightarrow dm = \lambda \, dx \Rightarrow dm = \frac{M}{L} \, dx$$

- M is the total mass of the rod
- L is the total length of the rod
- λ is the linear mass density of the rod, which is constant in this "uniform" rod.

y

dm

dx→ ←

←x→

L

Axis of Rotation (AOR)
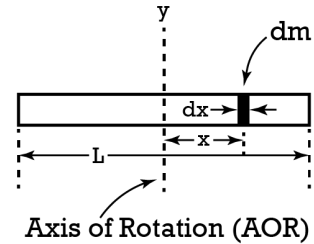
We can solve for the rotational inertia of the rod about its center of mass.

$$I_y = \int r^2 \, dm = \int r^2 \frac{M}{L} \, dx = \frac{m}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \, dx = \frac{M}{L} \left[\frac{x^3}{3} \right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$\Rightarrow I_y = \frac{M}{L} \left[\frac{\left(\frac{L}{2}\right)^3}{3} - \frac{\left(-\frac{L}{2}\right)^3}{3} \right] = \frac{M}{L} \left[\frac{L^3}{24} + \frac{L^3}{24} \right] = \frac{M}{L} \left[\frac{2L^3}{24} \right] = \boxed{\frac{1}{12} ML^2}$$

Moving the axis of rotation to the left end of the rod only changes the limits of the integral:

$$I_{end} = \frac{M}{L} \int_0^L x^2 \, dx \Rightarrow \frac{M}{L} \left[\frac{x^3}{3} \right]_0^L = \frac{M}{L} \left[\frac{L^3}{3} - \frac{0^3}{3} \right] = \boxed{\frac{1}{3} ML^2}$$

y

dm

dx→ ←

x→

L

Axis of Rotation (AOR)
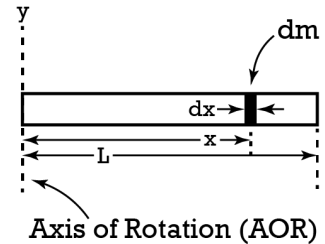
But now, we want to test our physics. In order to do so, we have the Rotational Inertia Demonstrator (RID) from Arbor Scientific. It is a pulley with three different pulley sizes and four long, thin, uniform density spokes which radiate from its center. Previously we measured the rotational inertia of the central pulley part which equals 0.00067 kg·m². Now we are going to use what we just learned to determine what the rotational inertia of the whole RID is:

$$I_{RID} = I_{pulley} + 4I_{spoke}$$

Therefore, we need to determine the rotational inertia of a single spoke of the RID. To do this, we are going to use the same integral as before, and change the limits.
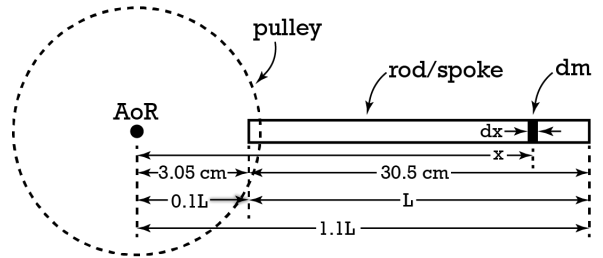
The length of one spoke is 30.5 cm and the end of the spoke starts 3.05 cm from the axis of rotation. Therefore the initial point is 0.1L:

$$x_i = \frac{3.05}{30.5}L = 0.1L$$

And the final point is 1.1 L:

$$x_f = \frac{3.05+30.5}{30.5}L = 1.1L$$

$$I_{spoke} = \frac{M}{L}\int_{0.1L}^{1.1L} x^2\,dx \Rightarrow \frac{M}{L}\left[\frac{x^3}{3}\right]_{0.1L}^{1.1L} = \frac{M}{L}\left[\frac{(1.1L)^3}{3} - \frac{(0.1L)^3}{3}\right] = \frac{M}{3L}\left[1.331L^3 - 0.001L^3\right]$$

$$\Rightarrow I_{spoke} = \frac{M}{3L}\left[1.33\overline{3}L^3\right] = 0.44\overline{3}ML^2 \approx \boxed{0.443ML^2}$$

Knowns: $M = 0.0742kg; L = 0.305m$

$$\Rightarrow I_{RID} = I_{pulley} + 4\left(0.44\overline{3}ML^2\right) = 0.00067 + (4)(0.44\overline{3})(0.0742)(0.305)^2 = 0.01292685$$

$$\Rightarrow I_{RID} \approx \boxed{0.0129kg\cdot m^2}$$

And we can test this. Previously we solved for the rotational inertia of an object in terms of the pulley radius, angular acceleration, and the force of tension acting on the pulley. See: "Graphing the Rotational Inertia of an Irregular Shape". https://www.flippingphysics.com/rotational-inertia-irregular-shape.html

$$\Delta t = 133 frames \times \frac{1sec}{60 frames} = 2.21\overline{6}\,sec; \Delta\theta = 2rev \times \frac{2\pi rad}{1rev} = 4\pi rad; \omega_i = 0$$

$$\alpha = \frac{2\Delta\theta}{\Delta t^2} = \frac{(2)(4\pi)}{(2.21\overline{6})^2} = 5.11492\frac{rad}{s^2}; R_{pulley} = 0.0286m; F_{T_{avg}} = 2.452N$$

$$I = \frac{R_{pulley}F_T}{\alpha} = \frac{(0.0286)(2.452)}{5.11492} = 0.013710 \approx 0.0137kg\cdot m^2$$

$$E_r = \frac{O-A}{A} \times 100 = \frac{0.13710 - 0.01292685}{0.01292685} \times 100 = 6.06075 \approx 6.06\%$$
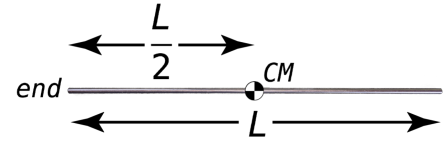
We have already derived the Parallel-Axis Theorem:[1]

$$I = I_{CM} + MD^2$$

We have already derived the equations for the rotational inertia of a uniform, long, thin rod about an axis which is perpendicular to the rod and goes through both its center of mass and its end:

$$I_{CM} = \frac{1}{12}ML^2 \ \& \ I_{end} = \frac{1}{3}ML^2$$

And we can use the parallel-axis theorem:

$$I_{end} = I_{CM} + MD^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + M\left(\frac{L^2}{4}\right) = \left(\frac{1}{12} + \frac{1}{4}\right)ML^2$$

$$\Rightarrow I_{end} = \left(\frac{1}{12} + \frac{3}{12}\right)ML^2 = \left(\frac{4}{12}\right)ML^2 \Rightarrow I_{end} = \frac{1}{3}ML^2$$

The Physics Works!!

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[1] http://www.flippingphysics.com/parallel-axis-theorem.html

Flipping Physics Lecture Notes:

2 Masses on a Pulley - Torque Demonstration

Example: 0.100 kg and 0.200 kg masses hang from either side of a frictionless pulley with a rotational inertia of 0.0137 kg·m² and radius of 0.0385 m. (a) What is the angular acceleration of the pulley? (b) What is the force of tension in each string?

Knowns:

$$m_1 = 0.100 kg; \ m_2 = 0.200 kg; \ I = 0.0137 kg \cdot m^2; \ R = 0.0385 m; \ \alpha = ?$$

Start with free body diagrams of the forces acting on the two masses and the pulley.

Note: If the pulley had no friction and no mass, the two forces of tension on either side of the pulley would be the same. In this example the pulley has mass and therefore rotational inertia, therefore, the forces of tension in the strings on either side of the pulley are not equal.

Let's start by summing the torques on the pulley with its axle as the axis of rotation. Because mass 2 is greater than mass 1, mass 2 should apply a larger torque on the pulley and cause the pulley to rotate in the clockwise, or into the board, direction. Therefore, let's define clockwise, or into the board, as positive. Notice that, because they both act on the axis of rotation, neither the force normal nor the force of gravity acting on the pulley will cause a torque on the pulley.

$$\sum \vec{\tau}_{pulley \atop AoR \ @ \ Axle} = \vec{\tau}_2 - \vec{\tau}_1 = I\vec{\alpha} \Rightarrow r_2 F_{T_2} \sin \theta_2 - r_1 F_{T_1} \sin \theta_1 = I\alpha$$

$$\Rightarrow RF_{T_2} \sin(90) - RF_{T_1} \sin(90) = I\alpha \Rightarrow RF_{T_2} - RF_{T_1} = I\alpha$$ (put in equation holster!)

We do not know either force of tension so we cannot currently solve for angular acceleration.

Sum the forces on mass 1:

$$\sum F_{mass \ 1 \atop + \ direction} = F_{T_1} - F_{g_1} = m_1 a_1 \Rightarrow F_{T_1} = F_{g_1} + m_1 a_1 = m_1 g + m_1 a_1$$ (put in equation holster!)

Sum the forces on mass 2:

$$\sum F_{mass \ 2 \atop + \ direction} = -F_{T_2} + F_{g_2} = m_2 a_2 \Rightarrow F_{T_2} = F_{g_2} - m_2 a_2 = m_2 g - m_2 a_2$$ (put in equation holster!)

Notice force of tension 2 may be *up,* however, according to the positive direction we defined, force of tension 2 is acting in the negative direction!

Combine all three equations!!!

$$I\alpha = RF_{T_2} - RF_{T_1} \ \& \ F_{T_1} = m_1 g + m_1 a_1 \ \& \ F_{T_2} = m_2 g - m_2 a_2$$

$$\Rightarrow I\alpha = R\left(m_2 g - m_2 a_2\right) - R\left(m_1 g + m_1 a_1\right)$$

Notice we can relate the linear accelerations to angular acceleration: $a_1 = a_2 = a_t = r\alpha = R\alpha$

$$\Rightarrow I\alpha = R\left(m_2 g - m_2 R\alpha\right) - R\left(m_1 g + m_1 R\alpha\right) = m_2 gR - m_2 R^2 \alpha - m_1 gR - m_1 R^2 \alpha$$
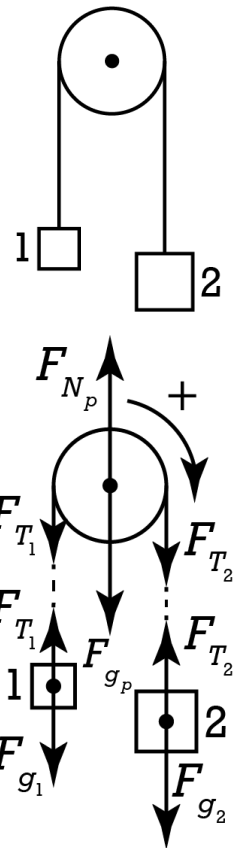
$$\Rightarrow I\alpha + m_2 R^2 \alpha + m_1 R^2 \alpha = m_2 gR - m_1 gR \Rightarrow \alpha\left(I + m_2 R^2 + m_1 R^2\right) = gR\left(m_2 - m_1\right)$$

$$\Rightarrow \alpha = \frac{gR\left(m_2 - m_1\right)}{I + m_2 R^2 + m_1 R^2} = \frac{gR\left(m_2 - m_1\right)}{I + R^2\left(m_2 + m_1\right)} = \frac{(9.81)(0.0385)(0.2 - 0.1)}{0.0137 + (0.0385)^2(0.2 + 0.1)} = 2.67016 \approx \boxed{2.67 \frac{rad}{s^2}}$$

Compare to the measured angular acceleration:

$$\omega_i = 0; \ \Delta\theta = 2rev \times \frac{2\pi rad}{1rev} = 4\pi rad; \ \Delta t = 193 frames \times \frac{1sec}{60 frames} = 3.21\bar{6}\sec; \ \alpha = ?$$

$$\Delta\theta = \omega_i \Delta t + \frac{1}{2}\alpha\Delta t^2 = \frac{1}{2}\alpha\Delta t^2 \Rightarrow \alpha = \frac{2\Delta\theta}{\Delta t^2} = \frac{(2)(4\pi)}{(3.21\bar{6})^2} = 2.4290 \approx 2.43 \frac{rad}{s^2}$$

$$E_r = \frac{O - A}{A} \times 100 = \frac{2.4290 - 2.67016}{2.67016} \times 100 = -9.0315 \approx -9.03\%$$

Now let's find the tensions:

$$F_{T_1} = m_1 g + m_1 a_1 = m_1\left(g + R\alpha\right) = (0.1)\left(9.81 + (0.0385)(2.67016)\right) = 0.991280 \approx \boxed{0.991N}$$

$$F_{T_2} = m_2 g - m_2 a_2 = m_2\left(g - R\alpha\right) = (0.2)\left(9.81 - (0.0385)(2.67016)\right) = 1.94144 \approx \boxed{1.94N}$$

Now, please notice that those two forces of tension are not equal in magnitude. Recall that is because our pulley has mass and therefore has a rotational inertia and therefore requires a net torque to angularly accelerated it. Therefore, in order to cause that net torque on the pulley, force of tension 1 and force of tension 2 cannot have the same magnitude. But if the pulley had negligible mass and therefore negligible rotational inertia, the equation we got from summing the net torques would actually show that the two forces of tension acting on the pulley would be the same.

$$RF_{T_2} - RF_{T_1} = I\alpha = (0)\alpha = 0 \Rightarrow RF_{T_2} = RF_{T_1} \Rightarrow F_{T_2} = F_{T_1}$$

I also want to point out two ways which students try to use to solve this problem which are both incorrect.

1) Sum the torques on the whole system all at once with the axis of rotation at the axle of the pulley. But realize, the rotational inertia and angular acceleration on the right hand side of the equation would then refer to everything in the system, including the two hanging masses. And hopefully you recognize that the hanging masses do not have rotational inertia nor do they have angular acceleration. So this equation is **incorrect**.

$$\sum \bar{\tau}_{\substack{everything \\ AoR @ Axle}} = \bar{\tau}_{F_{g_2}} - \bar{\tau}_{F_{T_2}} + \bar{\tau}_{F_{T_2}} - \bar{\tau}_{F_{T_1}} + \bar{\tau}_{F_{T_1}} - \bar{\tau}_{F_{g_1}} = I\alpha$$

2) Sum the forces on the entire system in the positive direction. This time it is because the mass times acceleration on the right hand side of the equation would be for the entire system. Now, I do understand that the rim of the pulley does have the same magnitude tangential acceleration as the linear accelerations of the two masses, however, the pulley itself does not have a tangential acceleration because tangential acceleration depends on radius. The larger the radius the larger the tangential acceleration of whatever specific point on the pulley you are referring to. So, this equation is also **incorrect**.

$$\sum \bar{F}_{\substack{everything \\ + direction}} = \bar{F}_{g_2} - \bar{F}_{T_2} + \bar{F}_{T_2} - \bar{F}_{T_1} + \bar{F}_{T_1} - \bar{F}_{g_1} = ma \ \& \ a_t = r\alpha$$

Example: Mass 1 and mass 2 hang from either side of a frictionless pulley with rotational inertia, I, and radius, R. What is the angular acceleration of the pulley?

Assume the system starts at rest and $m_2 > m_1$.

Knowns: $m_1, m_2, I, R, \alpha = ?$

There is neither a force applied or a force of friction which is adding or removing energy from the system, therefore, mechanical energy is conserved. Define the initial point as where the objects are when they are at rest and the final point as where the objects are after mass 1 has moved up a distance y and mass 2 has moved down the same distance y. Set the zero line at the initial point.

$$ME_i = ME_f \Rightarrow PE_{g_{2i}} = PE_{g_{1f}} + KE_{T_{1f}} + KE_{T_{2f}} + KE_{R_{pf}}$$

Initially the only type of mechanical energy is gravitational potential energy of mass 2. Finally, mass 1 has gravitational potential energy, both masses have translational kinetic energy, and the pulley has rotational kinetic energy. Note: The gravitational potential energy of the pulley is the same initially and finally, therefore it would have the same value on either side of the equal sign and would cancel out. That is why we have not included it.

Substituting in mechanical energy equations:

$$\Rightarrow m_2 g h_{2i} = m_1 g h_{1f} + \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 + \frac{1}{2} I \omega_{pf}^2$$

$$h_{2i} = h_{1f} = y \Rightarrow m_2 g y = m_1 g y + \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 + \frac{1}{2} I \omega_{pf}^2$$

Because they are attached by the string, the velocities of both masses are the same as the tangential velocity of the rim of the pulley: $v_t = r\omega \Rightarrow v_t = v_{1f} = v_{2f} = R\omega_{pf}$ (and multiply through by 2)

$$\Rightarrow 2 m_2 g y = 2 m_1 g y + m_1 \left(R\omega_{pf} \right)^2 + m_2 \left(R\omega_{pf} \right)^2 + I \omega_{pf}^2$$

$$\Rightarrow 2 m_2 g y = 2 m_1 g y + m_1 R^2 \omega_{pf}^2 + m_2 R^2 \omega_{pf}^2 + I \omega_{pf}^2$$

$$\Rightarrow 2 m_2 g y - 2 m_1 g y = \omega_{pf}^2 \left(m_1 R^2 + m_2 R^2 + I \right) \Rightarrow \omega_{pf}^2 = \frac{2 m_2 g y - 2 m_1 g y}{m_1 R^2 + m_2 R^2 + I}$$

Again, because they are attached by the string, both masses go through the same linear displacement, y,

$$s = r\Delta\theta \Rightarrow y = R\Delta\theta \Rightarrow \Delta\theta = \frac{y}{R}$$

which is the same as the arc length traveled by the rim of the pulley:

And the torque on the pulley will be constant so the pulley will experience constant angular acceleration, so we can use one of the uniformly angularly accelerated motion equations:
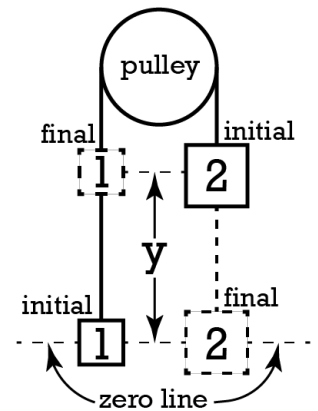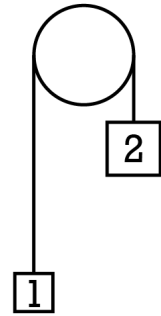
$$\omega_f^{\ 2} = \omega_i^{\ 2} + 2\alpha\Delta\theta = 2\alpha\Delta\theta \Rightarrow \alpha = \frac{\omega_f^{\ 2}}{2\Delta\theta} = \frac{R\omega_f^{\ 2}}{2y}$$ And then we can substitute in the equation we derived for the square of the final angular velocity of the pulley.

$$\Rightarrow \alpha = \left(\frac{R}{2y}\right)\left(\frac{2m_2 gy - 2m_1 gy}{m_1 R^2 + m_2 R^2 + I}\right) = \boxed{\frac{Rg(m_2 - m_1)}{R^2(m_1 + m_2) + I}}$$

This matches our answer from when we previously did this problem using Newton's Second Law, both the translation and rotational forms: https://www.flippingphysics.com/2-mass-pulley-torque.html

Torque - Mass on Plank with String

Example: A 0.300 kg mass rests on a 0.395 m long, 0.764 kg, uniform wooden plank supported by a string as shown in the figure. If the mass is 0.274 m from the wall and the angle between the string and the plank is 32.1°, (a) What is the force of tension in the string? and (b) What is the normal force from the wall?

The wood plank is at rest and not rotating, therefore it is in static equilibrium, therefore the net force acting on the plank equals zero and the net torque acting on the plank equals zero about any axis of rotation. The first thing we need to do is to draw the free body diagram of the forces acting on the plank.

Now that we have the free body diagram, we can sum the torques on the plank with an axis of rotation about the left end of the plank. Notice both the force normal and the force of static friction act at the axis of rotation and therefore cause no torque on the plank. (This is why we chose the left end as our axis of rotation. Notice how two out of our three unknown forces cause no torque with that axis of rotation. Helpful, eh?) We can make out counterclockwise, or out of the board, positive, therefore the torque caused by the force of tension is positive and the torques caused by the force of gravity of the plank and the force of gravity of the mass are both negative.
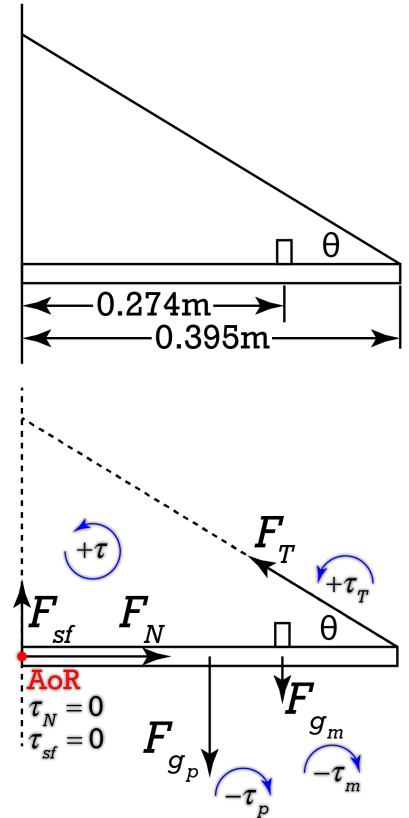
Knowns: $m_p = 0.764 kg$; $m_m = 0.300 kg$; $r_p = \dfrac{0.395m}{2} = 0.1975m$; $r_T = 0.395m$; $r_m = 0.274m$;

$\theta = 32.1°$; $(a) F_T = ?$ $(b) F_N = ?$

$$\sum \vec{\tau}_{plank \atop AoR@left end} = -\vec{\tau}_p - \vec{\tau}_m + \vec{\tau}_T = I\vec{\alpha} = I(0) = 0 \Rightarrow \vec{\tau}_T = \vec{\tau}_p + \vec{\tau}_m$$

$$\Rightarrow r_T F_T \sin\theta_T = r_p F_{g_p}\sin\theta_p + r_m F_{g_m}\sin\theta_m \Rightarrow r_T F_T \sin\theta_T = r_p m_p g \sin(90) + r_m m_m g \sin(90)$$

$$\Rightarrow F_T = \frac{r_p m_p g + r_m m_m g}{r_T \sin\theta_T} = \frac{(0.1975)(0.764)(9.81)+(0.274)(0.300)(9.81)}{(0.395)\sin(32.1)} = 10.8937 \approx \boxed{10.9N}$$

Next we are going to sum the forces in the x-direction, however, before we can, we need to break the force of tension in to its components. Actually, all we really need is the force of tension in the x-direciton.

$$\cos\theta_T = \frac{A}{H} = \frac{F_{T_x}}{F_T} \Rightarrow F_{T_x} = F_T \cos\theta_T$$
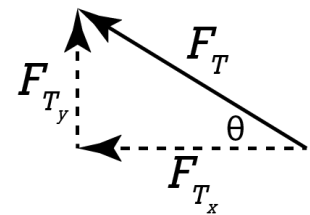
$$\sum F_x = F_N - F_{T_x} = ma_x = m(0) = 0 \Rightarrow F_N = F_{T_x} = F_T \cos\theta_T = (10.8937)\cos(32.1)$$

$$\Rightarrow F_N = 9.2283 \approx \boxed{9.23N}$$

P.s. I know we did not solve for the force of static friction (or, for that matter, the minimum coefficient of static friction to hold the plank on the wall), however, you are welcome to sum the forces in the y-direction, or pick a different axis of rotation and sum the torques on the plank, and you will be able to solve for that. Enjoy!

Realize these lecture notes will be much better understood with the visuals in the video at
https://www.flippingphysics.com/rolling-without-slipping.html

An object which is rolling without slipping stays in contact with the ground and does not slide relative to
the ground. Rolling without slipping combines translational with rotational motion.

Every part of the object in translational motion moves with the same velocity. We will define that as the

velocity of the center of mass of the translational object: $v_T = v_{cm}$

The center of mass of the rotational object has zero velocity: $v_{R\,cm} = 0$

The outer edge of the rotational object has a velocity equal to its tangential velocity: $v_{R\,edge} = R\omega$

The velocity of every point on the object rolling without slipping equals the addition of the tangential
velocity and the rotational velocity. Because the point of contact of the object rolling without slipping does
not move relative to the stationary ground, the velocity at the point of contact is zero. Therefore, the
velocity of the center of mass of an object rolling without slipping equals the radius of the object times the
angular velocity of the object:

$$\vec{v}_{bottom} = \vec{v}_T + \vec{v}_{R\,edge} \Rightarrow v_{bottom} = v_{cm} - R\omega = 0 \Rightarrow v_{cm} = R\omega$$

This is very much like the tangential velocity equation, however, "r" is the radius of the object: $v_t = r\omega$

The kinetic energy of an object rolling without slipping includes translational and rotational kinetic

energies: $$KE_{total} = KE_T + KE_R = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Notice the equations for distance travelled by and acceleration of an object rolling without slipping can
also be derived this same way and are also similar to their arc length and tangential acceleration

counterparts: $x_{cm} = R\Delta\theta \;\&\; s = r\Delta\theta; v_t = r\omega \;\&\; v_{cm} = R\omega; a_{cm} = R\alpha \;\&\; a_t = r\alpha$

Example: Determine the acceleration of a uniform, solid cylinder rolling without slipping down an incline with incline angle θ. The rotational inertia of a uniform, solid cylinder about its long cylindrical axis is $\frac{1}{2}MR^2$. Assume the cylinder starts from rest.

$a = ?$; Uniform Solid Cylinder; Rolling without Slipping; Incline Angle $= \theta$; $I = \frac{1}{2}MR^2$; $v_i = 0$

There is no work done by a force applied or a force of friction, therefore, there is no energy added or removed from this system, so mechanical energy is conserved. I do understand there is a force of static friction acting on the cylinder which causes it to rotate, however, because the cylinder does not slide, the force of friction does not cause any energy to be converted to heat or sound.

$$ME_i = ME_f$$

Set the initial point where the cylinder starts at rest and the final point after the cylinder has moved a distance $\Delta d_{\parallel}$. Set the zero line at the center of mass of the cylinder at the final position.
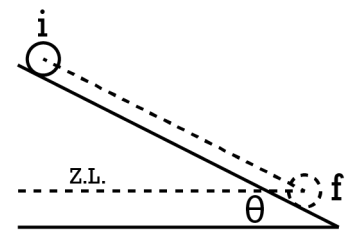
There is no spring so no elastic potential energy initial or final.
Initially the cylinder is at rest so no kinetic energy.
The only type of initial mechanical energy is gravitational potential energy because the cylinder is above the zero line.
There is not final gravitational potential energy because the cylinder is at the zero line.
The cylinder has translational kinetic energy final and rotational kinetic energy final because it is both rotating and its center of mass is translating from one point to another.

$$\Rightarrow mgh_i = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\omega_f^2$$
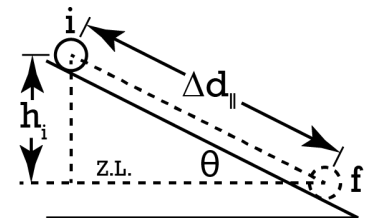
Substitute in the equation for rotational inertia of the solid cylinder.

The cylinder is rolling without slipping so $v_{cm} = R\omega = v_f \Rightarrow v_f^2 = R^2\omega_f^2$

$$\Rightarrow mgh_i = \frac{1}{2}mv_f^2 + \frac{1}{4}mv_f^2 \Rightarrow gh_i = \frac{1}{2}v_f^2 + \frac{1}{4}v_f^2 = \left(\frac{2}{4} + \frac{1}{4}\right)v_f^2 = \frac{3}{4}v_f^2 \Rightarrow v_f^2 = \frac{4}{3}gh_i$$

Everybody brought mass to the party!

$$\sin\theta = \frac{O}{H} = \frac{h_i}{\Delta d_{\parallel}} \Rightarrow h_i = \Delta d_{\parallel}\sin\theta \Rightarrow v_f^2 = \frac{4}{3}g\Delta d_{\parallel}\sin\theta$$

Acceleration down the incline will be constant, so we can use the uniformly accelerated motion equation:

$$v_{f_{\parallel}}^2 = v_{i_{\parallel}}^2 + 2a_{\parallel}\Delta d_{\parallel} \Rightarrow \frac{4}{3}g\Delta d_{\parallel}\sin\theta = 2a_{\parallel}\Delta d_{\parallel} \Rightarrow a_{\parallel} = \left(\frac{1}{2}\right)\left(\frac{4}{3}g\sin\theta\right) = \boxed{\frac{2}{3}g\sin\theta}$$

Notice this means the only variables which affect the acceleration of a uniform object rolling without slipping down an incline are the planet (acceleration due to gravity), the incline angle θ, and the shape of the object. What I mean by the "shape of the object" is the factor in front of the MR^2 in the rotational inertia equation. Not the mass or radius of the object, but rather just that factor in front of the MR^2.

Demonstration: $g = 9.81 \dfrac{m}{s^2}; \theta = 15.3° \Rightarrow a_{\parallel} = \dfrac{2}{3}(9.81)\sin(15.3) = 1.72573 \approx 1.73 \dfrac{m}{s^2}$


y = 1.73x - 0.01
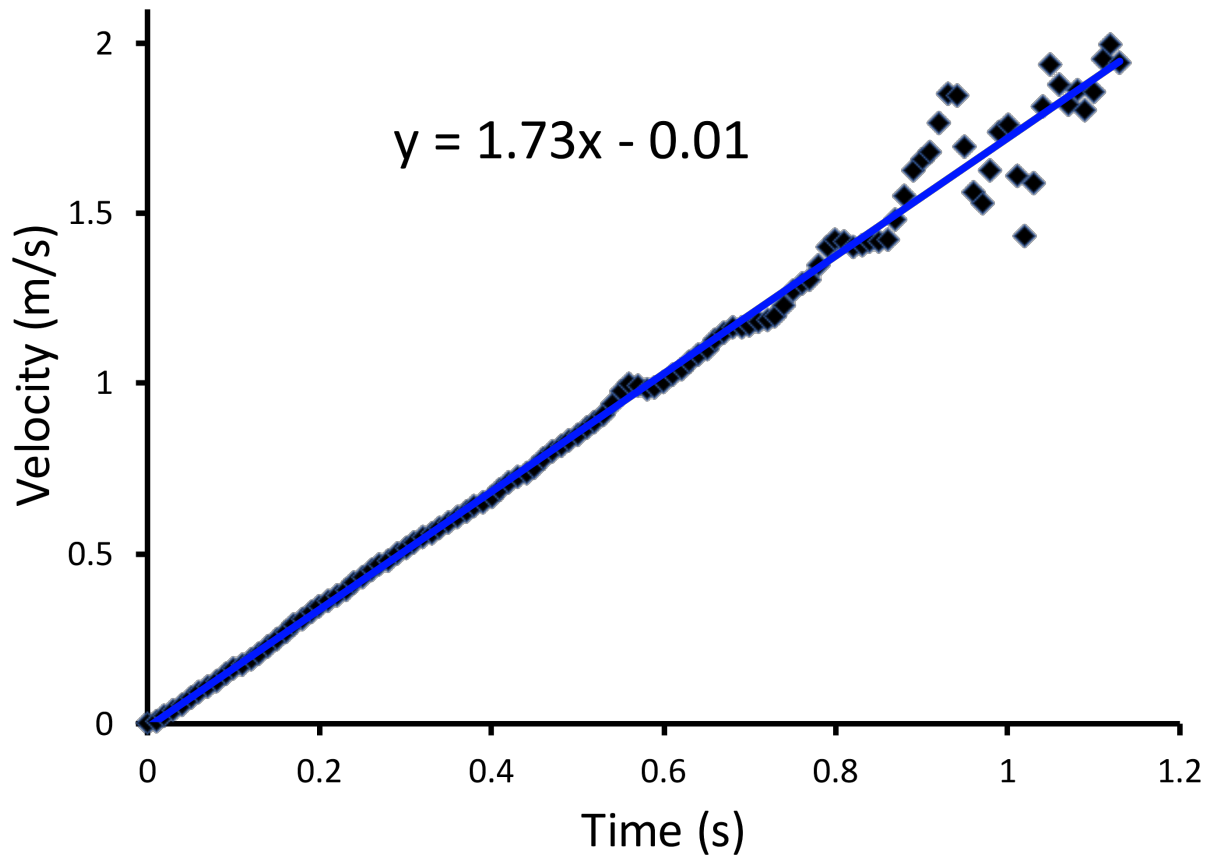
The slope of velocity a function of time is acceleration, so our experimental acceleration down the incline is also 1.73 m/s²!!

Which Will Be First? (Rolling Down an Incline)

Example: A hollow sphere, solid sphere, and thin hoop are simultaneously released from rest at the top of an incline. Which will reach the bottom first? Assume all objects are of uniform density.
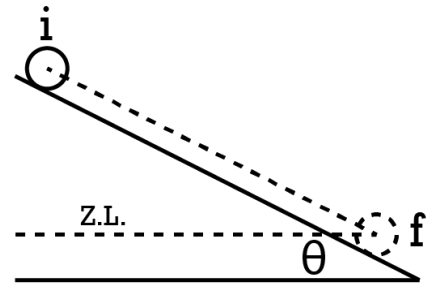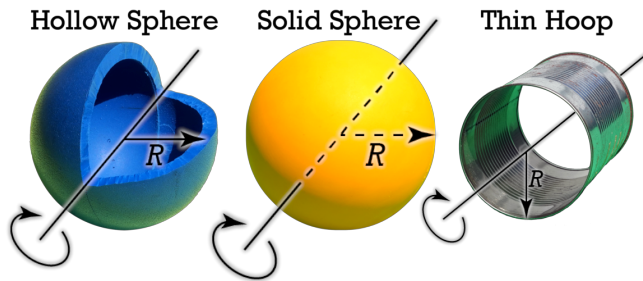
According to conservation of mechanical energy, each object starts with all gravitational potential energy and ends with translational and rotational kinetic energies.

$$ME_i = ME_f \Rightarrow U_{g_i} = KE_{T_f} + KE_{R_f}$$

In our previous lesson we showed that only the acceleration due to gravity, incline angle, and fraction in front of the MR^2 equation for rotational inertia of the object affect the acceleration down the incline. All three objects are on the same incline and planet, therefore the only difference for each object is the fraction in front of the MR^2 equation for rotational inertia.

Considering the equation for rotational kinetic energy is $KE_R = \dfrac{1}{2}I\omega^2$, a smaller rotational inertia, I, will mean a smaller rotational kinetic energy. A smaller rotational kinetic energy will mean more energy left over for translational kinetic energy. In other words:

$$I\downarrow \Rightarrow KE_R\downarrow \Rightarrow KE_T\uparrow \Rightarrow v_f\uparrow \Rightarrow \Delta t\downarrow$$

So the smallest fraction for the rotational inertia equation for the object will mean the largest final velocity, which means it will get there first.

Rather than looking up the rotational inertia equations for solid sphere, hollow sphere, and thin hoop, realize you should be able to compare their relative MR^2 fractions empirically using the equation for rotational inertia:

$$I_{\substack{system\,of\\particles}} = \sum_i m_i r_i^2$$

In other words, the more mass an object has located farther from its axis of rotation, the higher the rotational inertia. Therefore, the order of fraction, X, in the rotational inertia equation should be:

$$I = X\left(MR^2\right) \,\&\, X_{solid\,sphere} < X_{hollow\,sphere} < X_{thin\,hoop}$$

Therefore, the order of final velocities should be:

$$v_{f_{solid\,sphere}} > v_{f_{hollow\,sphere}} > v_{f_{thin\,hoop}} \Rightarrow \Delta t_{solid\,sphere} < \Delta t_{hollow\,sphere} < \Delta t_{thin\,hoop}$$

Therefore, the order of objects reaching the bottom of the incline should be:
- 1st: Solid Sphere
- 2nd: Hollow Sphere
- 3rd: Thin Hoop

Which our demonstration clearly shows to be true. ☺

Example: A rope is wrapped around a bicycle wheel. The wheel is released from rest and allowed to descend without slipping as the rope unwinds from the wheel. While descending, does the center of the wheel move straight down, toward the left, or toward the right?
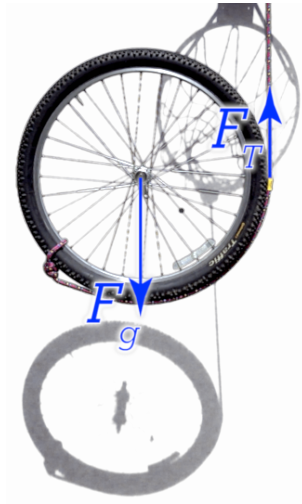
$$\sum F_x = 0 = ma_x \Rightarrow a_x = 0$$ No acceleration in the x direction.

$$\sum F_y = ma_y \neq 0 \Rightarrow a_y \neq 0$$ There is an acceleration in the y direction.

Center of the wheel will accelerate downward and not to the left or to the right.

Example: A rope is wrapped around a bicycle wheel with a rotational inertia of $0.68MR^2$. The wheel is released from rest and allowed to descend without slipping as the rope unwinds from the wheel. In terms of g, determine the acceleration of the wheel as it descends.

$$\sum F_y = F_T - F_g = ma_y \Rightarrow F_T = F_g + Ma_y = Mg + Ma_y \Rightarrow F_T = M\left(g + a_y\right)$$

$$\sum \tau_{\substack{AoR\ at\ Center \\ of\ Wheel}} = -\tau_T = I\alpha \Rightarrow -rF\sin\theta = -RF_T\sin 90 = \left(0.68MR^2\right)\left(\frac{a_y}{R}\right)$$

$$a_t = r\alpha \Rightarrow a_{cm} = R\alpha = a_y \Rightarrow \alpha = \frac{a_y}{R}$$

Note:

$$\Rightarrow F_T = -0.68Ma_y = M\left(g + a_y\right) \Rightarrow -0.68a_y = g + a_y \Rightarrow -1.68a_y = g \Rightarrow a_y = -\frac{g}{1.68}$$

$$\Rightarrow a_y = -0.59524g \approx \boxed{-0.60g}$$

Testing our answer:

$$a_y = -0.59524g = -\left(0.59524\right)\left(9.81\right) = -5.8393 \approx -5.8\frac{m}{s^2}$$

Predicted acceleration

$$\Delta y = -0.75m;\ v_{iy} = 0;\ \Delta t = 0.49\sec$$

$$\Delta y = v_{iy}\Delta t + \frac{1}{2}a\Delta t^2 = \frac{1}{2}a\Delta t^2 \Rightarrow a = \frac{2\Delta y}{\Delta t^2} = \frac{\left(2\right)\left(-0.75\right)}{0.49^2} = -6.2474 \approx -6.2\frac{m}{s^2}$$

Measured acceleration

Example: As shown, a rope is wrapped around a bicycle wheel with a rotational inertia of $0.68MR^2$. The wheel is released from rest and allowed descend without slipping as the rope unwinds from the wheel. In terms of g, determine the acceleration of the wheel as it unwinds from the wheel.[1]
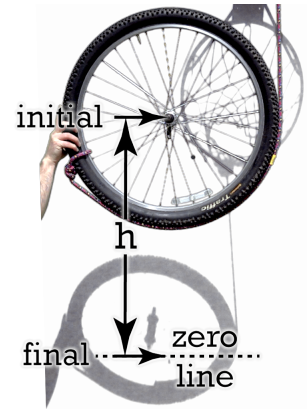
Conservation of Mechanical Energy with initial point where the wheel starts, final point after the wheel has gone down a distance h, and horizontal zero line at the final point.

$$ME_i = ME_f \Rightarrow mgh_i = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2$$

There are no springs in the problem, so no elastic potential energy initial or final.
Initially the wheel is at rest, so it has no initial kinetic energy.
The wheel is initially above the zero line, so it has initial gravitational potential energy.
At the final point the wheel is at the zero line, so it has no final gravitational potential energy.
At the final point the wheel is both rotating and moving translationally, so it has both final translational kinetic energy and final rotational kinetic energy.

$$\Rightarrow \cancel{M}gh = \frac{1}{2}\cancel{M}v_f^2 + \frac{1}{2}\left(0.68\cancel{M}R^2\right)\omega_f^2$$

The mass of the wheel is M, initial height of the wheel is h, and the rotational inertia of the wheel is $0.68MR^2$. Everybody brought mass to the party so we can be equitable and take mass from everybody.

$$v_t = r\omega \Rightarrow v_{cm} = R\omega_f = v_f \Rightarrow v_f^2 = R^2\omega_f^2$$

Remember the velocity of the center of mass of an object which is rolling without slipping is similar to the tangential velocity equation. The bicycle wheel is rolling without slipping down the rope, so the velocity of the center of mass of the wheel is the final velocity of the wheel.

$$\Rightarrow gh = \frac{1}{2}v_f^2 + 0.34R^2\omega_f^2 = \frac{1}{2}v_f^2 + 0.34v_f^2 = 0.84v_f^2 \Rightarrow v_f^2 = \frac{gh}{0.84}$$

We can substitute v_f^2 in for $R^2\omega_f^2$ in our equation and solve for v_f^2.

The wheel has a uniform acceleration, so we can use one of the uniformly accelerated motion equations:

$$v_f^2 = v_i^2 + 2a_y\Delta y \Rightarrow \frac{\cancel{gh}}{0.84} = 2a_y\left(-\cancel{h}\right) \Rightarrow \frac{g}{0.84} = -2a_y$$

The initial velocity of the wheel is zero, substitute $\frac{gh}{0.84}$ in for v_f^2, and substitute $-h$ in for Δy because the wheel goes down a distance "h" from the initial to the final points. Everybody brought "h" to the party… And now we can solve for the acceleration of the wheel in the y-direction:

$$a_y = -\frac{1}{2}\left(\frac{g}{0.84}\right) = -0.59524g \approx \boxed{-0.60g}$$

(The negative means the acceleration is down.)

Our answer matches when we did this using torque instead of energy! ☺

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[1] Previously we did this problem using net force and net torque instead of conservation of mechanical energy.
https://www.flippingphysics.com/wheel-rope-acceleration-torque.html

We already know the equation for the linear momentum of an object: $\vec{p} = m\vec{v}$

An object can also have angular momentum: $\vec{L} = I\vec{\omega}$
- The symbol for angular momentum is capital L.
- Linear momentum has inertial mass; angular momentum has rotational inertia.
- Linear momentum has linear velocity; angular momentum has angular velocity.
- Angular momentum is a vector and its direction is the same as the direction of the angular velocity of the object.
- Angular momentum has an **axis of rotation!** (which you must identify)
- The units for angular momentum are:
  $$\vec{L} = I\vec{\omega} \Rightarrow \left(kg \cdot m^2\right)\left(\frac{rad}{s}\right) \Rightarrow \frac{kg \cdot m^2}{s}$$

$\vec{L} = I\vec{\omega}$ is the equation for the angular momentum of Rigid Objects with Shape. In other words, objects which do not change shape easily and are larger than point particles. For example, disks, cylinders, spheres, planets, etc.

Example: Determine the angular momentum of a 141 g, 31.4 cm diameter record rotating clockwise at 45 revolutions per minute. $I_{disc} = \frac{1}{2}MR^2$

Knowns: $m = 141g \times \dfrac{1kg}{1000g} = 0.141kg;\ Dia = 31.4cm;\ R = \dfrac{31.4}{2} = 15.7cm \times \dfrac{1m}{100cm} = 0.157m;$

$\omega = -45\dfrac{rev}{min} \times \dfrac{1min}{60\sec} \times \dfrac{2\pi rad}{1rev} = -4.7124\dfrac{rad}{s};\ I_{disk} = \dfrac{1}{2}MR^2;\ L = ?$ Axis of rotation at Center of Mass of Disk.

$\vec{L} = I\vec{\omega} = \dfrac{1}{2}MR^2\vec{\omega} = \dfrac{1}{2}(0.141)(0.157)^2(-4.7124) = -0.00818899 \approx -0.0082\dfrac{kg \cdot m^2}{s}$

But what about the direction? According to the right-hand rule, the fingers of your right hand curl in the direction the record is rotating and you stick out the thumb of your right hand which points in the direction of the angular velocity and angular momentum, which is into the page, which is negative.

$$\vec{L} \approx -0.00819\frac{kg \cdot m^2}{s}$$

We have already learned about conservation of linear momentum: $\sum \vec{p}_i = \sum \vec{p}_f$

As you can imagine, the equation for conservation of angular momentum is similar: $\sum \vec{L}_i = \sum \vec{L}_f$

We will talk about when angular momentum is conserved in a little bit, let's start however, with a simple demonstration of conservation of angular momentum.

If I am spinning on a stool holding two masses at arm's length straight out from my body and then pull in my arms, what will happen? From general knowledge of sports like figure skating, gymnastics, dance, and diving, you probably already know that my angular velocity will increase. It is important to know why. The equation for angular momentum of a rigid object with shape is: $\vec{L} = I\vec{\omega}$. Therefore, the equation for conservation of angular momentum in the example of me spinning on a stool is:

$$\sum \vec{L}_i = \sum \vec{L}_f \Rightarrow I_i\vec{\omega}_i = I_f\vec{\omega}_f$$

It is important to recognize that the axis of rotation for the angular momenta and rotational inertias in these equations is the vertical axis through the center of the stool.

$$I_{particles} = \sum_i m_i r_i^2$$

The equation for the rotational inertia of a system of particles is:
In other words, bringing in my arms decreases the average distance the particles of the system are from the axis of rotation, which decreases the rotational inertia of the system, therefore, because angular momentum is conserved, the angular velocity of the system must increase.

$$r \downarrow \Rightarrow I \downarrow \Rightarrow \vec{\omega} \uparrow$$

Now let's talk about when angular momentum is conserved. As you recall linear momentum is conserved when the net force on the system equals zero.

$$\sum \vec{F}_{\substack{external \\ on\ system}} = 0 = \frac{\Delta \vec{p}_{system}}{\Delta t} \Rightarrow 0 \cdot \Delta t = \left(\frac{\Delta \vec{p}_{system}}{\Delta t} \right) \Delta t \Rightarrow 0 = \Delta \vec{p}_{system} = \vec{p}_{f\,system} - \vec{p}_{i\,system} \Rightarrow \vec{p}_{i\,system} = \vec{p}_{f\,system}$$

Angular momentum is conserved when the net external torque acting on the system equals zero.

$$\sum \vec{\tau}_{\substack{external \\ on\ system}} = 0 = \frac{\Delta \vec{L}_{system}}{\Delta t} \Rightarrow 0 \cdot \Delta t = \left(\frac{\Delta \vec{L}_{system}}{\Delta t} \right) \Delta t \Rightarrow 0 = \Delta \vec{L}_{system} = \vec{L}_{f\,system} - \vec{L}_{i\,system} \Rightarrow \vec{L}_{i\,system} = \vec{L}_{f\,system}$$

Hopefully you can see the similarities in the derivations. ☺

Returning back to me sitting on the stool. Notice that, about the vertical axis through the center of the stool, the net external torque acting on the system of me and the stool is zero, therefore angular momentum of the system will stay constant. The initial angular velocity of the system is zero, therefore, I can wave my arms around all I want, but doing so will not change the angular momentum of the system. However, if I push on something external to the system, I can cause a net torque on the system, angular momentum is no longer conserved, and I can increase my angular velocity.

Remember that angular momentum is a vector, therefore, when angular momentum is conserved, its direction is conserved as well. This is why a spinning top will maintain its vertical position. Its angular momentum, according to the right hand rule, will be vertical and therefore, as long as the top continues to spin, the angular momentum will be conserved and the top will stay vertical. However, a top which is not spinning, has no angular momentum, and will not stay vertical.

We can also apply this concept to a moving bicycle. The wheels of the bike, while the bike is moving, are spinning and have angular momentum. While you are moving forward, the direction of the angular momentum of the wheels will be to the left. Conservation of angular momentum will try to maintain the direction of the angular momentum of the wheels and therefore will help keep the bicycle vertical. If the bike is not moving, the wheels have no angular momentum and therefore do not help keep the bicycle vertical. Conservation of angular momentum is why it is easier to balance on a bike while it is moving.

Wheel Conservation of Angular Momentum Demonstration and Solution

Let's start with me sitting on a stool free to rotate around a vertical axis through the center of the stool. About that axis, there is no net external torque acting on the stool and me system, so angular momentum of the system is conserved. Initial momentum of the system is zero, so as much as I wave my arms and legs around, I cannot cause the stool to rotate because the angular momentum of the system will stay zero.

However, if I am holding a spinning wheel, the wheel has angular momentum and if I apply a torque to the spinning wheel, the spinning wheel will apply an equal but opposite torque back on me which will cause me to rotate. Note, the angular momentum of the system is still conserved because the two torques, which are equal and opposite, will cancel out, and result in a net torque on the system of zero. In other words, both of these torques, the torque I apply to the wheel and the equal but opposite torque the wheel applies on me, are internal to the person, chair, and wheel system.

Example: A person is holding a spinning wheel while sitting on a stool which is free to rotate. If the person rotates the wheel 180° about a horizontal axis, in terms of the initial angular momentum of the wheel, what is the final angular momentum of the person and stool? (assume no friction)

$$\sum \vec{L}_i = \sum \vec{L}_f; \vec{L} = I\vec{\omega} \ \& \ \vec{\omega}_{i\,stool+person} = 0 \Rightarrow \vec{L}_{i\,stool+person} = 0;$$

$$\vec{L}_{f\,stool+person} = X\vec{L}_{i\,wheel}; X = ?; \vec{L}_{f\,wheel} = -\vec{L}_{i\,wheel}$$

$$\sum \vec{L}_i = \sum \vec{L}_f \Rightarrow \vec{L}_{i\,wheel} + \vec{L}_{i\,stool+person} = \vec{L}_{f\,wheel} + \vec{L}_{f\,stool+person}$$

$$\Rightarrow \vec{L}_{i\,wheel} + 0 = -\vec{L}_{i\,wheel} + \vec{L}_{f\,stool+person} \Rightarrow \boxed{\vec{L}_{f\,stool+person} = 2\vec{L}_{i\,wheel}}$$

There is an interesting result to our answer. Notice the direction of the final angular momentum of the stool and person is in the same direction as the initial angular momentum of the wheel. This is true regardless of which direction the person rotates the spinning wheel. ☺

Just so you know, this same physics is how NASA rotates the Hubble Space Telescope. In space there is nothing to push off of to cause a net external torque on the telescope. Therefore, there are near frictionless gyroscopes constantly spinning on the telescope which can be rotated, which in turn rotates the direction the Hubble Space Telescope is pointing. The International Space Station also has gyroscopes to control its rotation. Which is pretty darn cool, if you ask me.

Merry-Go-Round
Conservation of Angular Momentum Problem

Example: A 25 kg child is sitting on the edge of a merry-go-round. The merry-go-round has a mass of 255 kg and is rotating at 2.0 radians per second. The child crawls to the middle of the merry-go-round. What is the final angular speed of the merry-go-round? You may make the following estimations: The child is a point particle; the merry-go-round is a solid disk and has an axle with negligible friction. $I_{disk} = \frac{1}{2}MR^2$.

Knowns: $m_c = 25 kg$; $m_w = 255 kg$; $\omega_{wi} = 2.0 \dfrac{rad}{s}$; child: outside edge $\rightarrow$ center; $I_{disk} = \dfrac{1}{2}MR^2$;

child = point particle; $\mu_w \approx 0$; $\omega_{wf} = ?$

For subscripts let's use c for child and w for wheel (a.k.a merry-go-round). Because net torque equals change in angular momentum over change in time, we know angular momentum is conserved when the net external torque acting on the system equals zero. There is no friction on the axle of the merry-go-round, so, because it equals zero, friction on the axle does not cause an external torque on the merry-go-round. As the child crawls toward its center, there is a force of static friction from the merry-go-round on the child and, according to Newton's third law, an equal but opposite force of static friction from the child on the merry-go-round. Because the axis of rotation is at the center of the merry-go-round, those two forces cause equal but opposite torques on the child and merry-go-round system, which means the two torques are internal to the system and cancel one another out. So the net torque on the system is zero and angular momentum, about the axle of the merry-go-round, is conserved.

$$\sum \vec{\tau}_{external} = \frac{\Delta \vec{L}}{\Delta t} = 0 \Rightarrow \sum \vec{L}_i = \sum \vec{L}_f$$

- There is no direction given in the problem and we are solving for angular speed. So let's drop the vector symbol from the equations and solve for the magnitude of final angular velocity.

- The equation for angular momentum of a rigid object with shape is: $\vec{L} = I\vec{\omega}$.

- When a point particle is moving in circular motion, the point particle has an angular momentum: $\vec{L} = I\vec{\omega}$. We will assume this to be true for now, however, we will prove this in a later lesson.

$$\sum \vec{L}_i = \sum \vec{L}_f \Rightarrow L_{ci} + L_{wi} = L_{cf} + L_{wf} \Rightarrow I_{ci}\omega_{ci} + I_w\omega_{wi} = I_{cf}\omega_{cf} + I_w\omega_{wf}$$

- Notice that we do not need subscripts for initial and final for the rotational inertia of the wheel/merry-go-round because its rotational inertia does not change.

$$I_{\substack{system\,of \\ particles}} = \sum_i m_i r_i^2 \Rightarrow I_c = m_c r_c^2$$

- Both objects have the same angular velocities: $\omega_{ci} = \omega_{wi} = \omega_i$ & $\omega_{cf} = \omega_{wf} = \omega_f$

- For the radius of the wheel, let's use R: $r_w = R$

$$\Rightarrow \left(m_c r_{ci}^2\right)\omega_i + \left(\frac{1}{2}m_w R^2\right)\omega_i = \left(m_c r_{cf}^2\right)\omega_f + \left(\frac{1}{2}m_w R^2\right)\omega_f$$

- The initial distance from the axis of rotation to the location of the child is the same as the radius of the wheel: $r_{ci} = R$.
- The final distance from the axis of rotation to the location of the child is zero: $r_{cf} = 0$.
  - Realize this means our child has zero rotational inertia when sitting at the center of the merry-go-round. Hopefully you realize this will not be true in real life because the child has non-zero size and therefore will have rotational inertia. This is a simplified solution and helps with understanding. The child's small size and therefore quite small rotational inertia relative to the merry-go-round makes this an okay estimation.

$$\Rightarrow \cancel{m_c R^2} \omega_i + \left(\frac{1}{2} m_w \cancel{R^2}\right)\omega_i = \cancel{m_c (0)^2} \omega_f + \left(\frac{1}{2} m_w \cancel{R^2}\right)\omega_f$$

- Everybody brought R^2 to the party!

$$\Rightarrow m_c \omega_i + \frac{1}{2} m_w \omega_i = \frac{1}{2} m_w \omega_f \Rightarrow 2m_c \omega_i + m_w \omega_i = m_w \omega_f \Rightarrow \omega_f = \frac{\omega_i \left(2m_c + m_w\right)}{m_w}$$

$$\omega_f = \frac{(2)\left[(2)(25)+255\right]}{255} = 2.39216 \approx \boxed{2.4 \frac{rad}{s}}$$

Does it make sense that the angular velocity of the system increases as the child moves toward the middle of the merry-go-round? Let's go back to the conservation of angular momentum equation near the beginning of the solution:

$$\Rightarrow I_{ci}\omega_{ci} + I_w \omega_{wi} = I_{cf}\omega_{cf} + I_w \omega_{wf} \Rightarrow I_{ci}\omega_i + I_w \omega_i = I_{cf}\omega_f + I_w \omega_f \Rightarrow \left(I_{ci} + I_w\right)\omega_i = \left(I_{cf} + I_w\right)\omega_f$$

Angular momentum is conserved. The rotational inertia of the wheel remains unchanged throughout the whole event. But as the child moves in toward the axis of rotation, the rotational inertia of the child decreases, therefore, in order to maintain a constant angular momentum of the system, the angular velocity of the system has to increase. In other words, the angular momentum of the system does not change, so if the rotational inertia of the system decreases, the angular velocity of the system must increase.

Realize, this increase in angular velocity of the system actually represents an increase in the kinetic energy of the system. There is no translational motion of the system, so all of the kinetic energy is rotational kinetic energy. The change in kinetic energy of the system is:

$$\Delta KE = KE_f - KE_i = KE_{cf} + KE_{wf} - KE_{ci} - KE_{wi}$$

$$\Rightarrow \Delta KE = \frac{1}{2}I_{cf}\omega_f^2 + \frac{1}{2}I_w \omega_f^2 - \frac{1}{2}I_{ci}\omega_i^2 - \frac{1}{2}I_w \omega_i^2 = \frac{1}{2}(0)\omega_f^2 + \frac{1}{2}\left(\frac{1}{2}m_w R^2\right)\omega_f^2 - \frac{1}{2}\left(m_c R^2 + \frac{1}{2}m_w R^2\right)\omega_i^2$$

$$\Rightarrow \Delta KE = R^2 \left[\frac{1}{4}m_w \omega_f^2 - \left(\frac{1}{2}m_c + \frac{1}{4}m_w\right)\omega_i^2\right]$$

$$\Rightarrow \Delta KE = R^2 \left[\frac{1}{4}(255)(2.39216)^2 - \left(\frac{1}{2}25 + \frac{1}{4}255\right)(2)^2\right] = 59.804 R^2 \approx 6.0 \times 10^1 R^2 = W_{net}$$

The change in kinetic energy of the system is positive because the child had to do work on herself, and therefore the system, in order to crawl from the outside edge to the center of the merry-go-round. Because net work equals change in kinetic energy, we know she did $60R^2$ joules of work on the system to increase the kinetic energy of the system. The larger the radius of the merry-go-round, the more work she has to do to the system.

We have already worked with the angular momentum of rotating objects like records, spinning tops, and merry-go-rounds. Because those objects are rotating, it is probably clear they have angular momentum. Today we are going to show that an object which is moving in a straight line can also have angular momentum.

You really need to watch the video at http://www.flippingphysics.com/angular-momentum-particles.html

The equation for angular momentum of a rigid object with shape is: $\vec{L} = I\vec{\omega}$

Because the wooden board is not moving before the collision with the point particle, the board has an angular velocity of zero and therefore an angular momentum of zero before the collision. After the collision, the wooden board does have an angular velocity and therefore angular momentum. Because the wooden board has gained angular momentum, that angular momentum must have come from somewhere. The explanation is that the point particle, which was moving in a straight line, transferred some of its angular momentum to the wooden board. Therefore, the point particle, which was moving in a straight line, has angular momentum.
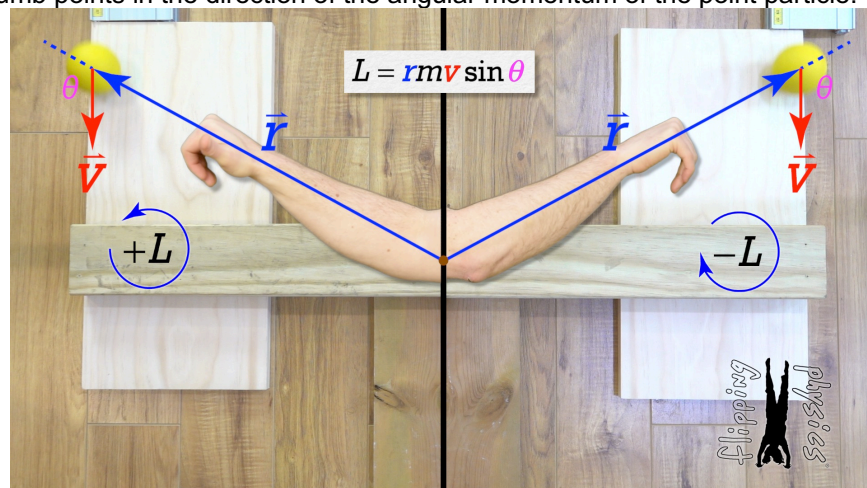
The equation for the magnitude of the angular momentum of a point particle is: $L = rmv \sin\theta$
In the video we give evidence that this equation is correct.
- r is the vector pointing from the axis of rotation to the center of mass of the point particle.
- m is the mass of the point particle.
- v is the velocity of the point particle.
- θ is the angle between the direction of the r vector and the velocity of the point particle.

Please realize angular momentum is still about an axis of rotation and you need to identify the axis of rotation when using the equation for the angular momentum of a point particle.

Angular momentum is a vector and the way you determine the direction of the angular momentum of a point particle is by using the right hand rule. Fingers of your right hand start at the axis of rotation, point your fingers in the direction of "r", curl your fingers in the direction of the velocity, and stick out your thumb. Your thumb points in the direction of the angular momentum of the point particle.

In our previous lesson we used a wooden board mounted on a Lazy Susan to introduce the equation for the angular momentum of a point particle: $L = rmv \sin\theta$

- r is the vector pointing from the axis of rotation to the center of mass of the point particle.
- m is the mass of the point particle.
- v is the velocity of the point particle.
- θ is the angle between the direction of the r vector and the velocity of the point particle.

In this lesson we are going to introduce a common triangle used when a point particle is moving towards a rigid object with shape. In this example, the point particle is moving toward the wooden board and will collide with the board. During that collision, angular momentum is conserved and an important piece to understand is what the angular momentum of the point particle is *before* it strikes the wooden board.
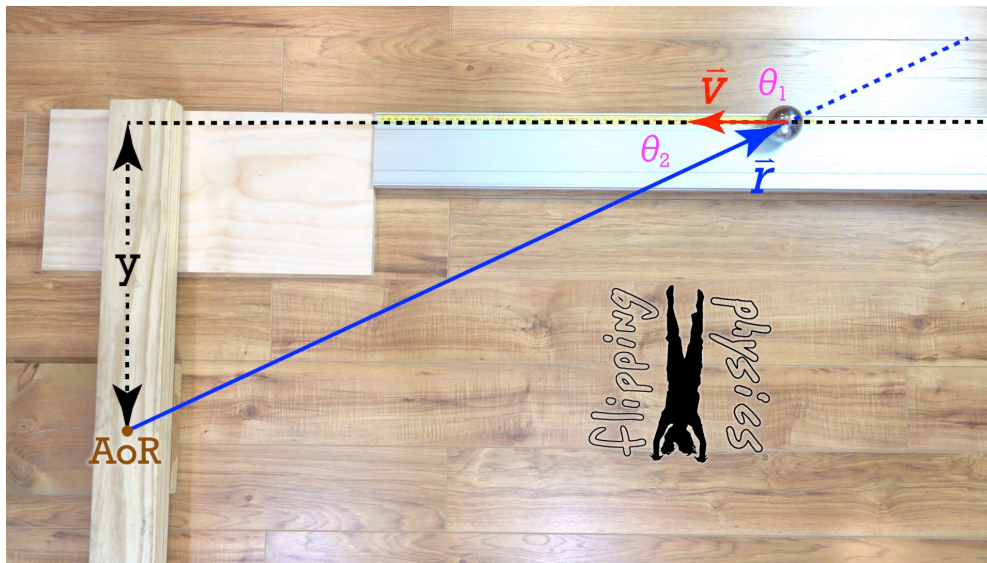
The angle in the angular momentum equation is the angle between the direction of the r vector and the velocity vector, so it is θ₁ however, because θ₁ and θ₂ are supplementary angled (they add up to 180°) the sines of those two angles are equal:

$$\theta_1 + \theta_2 = 180° \Rightarrow \sin\theta_1 = \sin\theta_2 \; (\textbf{supplementary angles})$$

Therefore, when using the angular momentum equation, we can use either angle.

$$\sin\theta = \frac{O}{H} = \frac{y}{r} \Rightarrow y = r\sin\theta = \text{constant}$$

This means while both r and θ change as a function of position, the distance y is constant. That means that, if the point particle is moving at a constant velocity, the angular momentum of the point particle is also constant.

$$L = rmv \sin\theta = mvy = \text{constant} \; \big(\text{when } v = \text{constant}\big)$$

This basic concept is used frequently when solving conservation of angular momentum problems involving a point particle and a rigid object with shape.

Flipping Physics Lecture Notes:

What are the Equations for
Kinetic Energy and Angular Momentum
of a Point Particle Moving in a Circle?

The equation for the Kinetic Energy of a translationally moving object:
$$KE_T = \frac{1}{2}mv^2$$

The equation for the Kinetic Energy of a rotating object is:
$$KE_R = \frac{1}{2}I\omega^2$$

A point particle which is moving in a circle is moving translationally because its center of mass is not in one constant location and it is rotating because it is moving around an axis of rotation. So which equation for Kinetic Energy should we use?

The equation for the rotational inertia of a system of particles is:
$$I_{\substack{system\ of\\particles}} = \sum_i m_i r_i^2$$

Therefore, the equation for the rotational inertia of a single particle is: $I_{point\ particle} = mR^2$
Where R is defined as the radius of the circle the point particle is describing.

This means the Rotational Kinetic Energy is:
$$KE_R = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(mR^2\right)\omega^2$$

And the linear velocity of the point particle is a tangential velocity: $v_t = r\omega \Rightarrow v^2 = R^2\omega^2$

Therefore:
$$KE_R = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(mR^2\right)\omega^2 = \frac{1}{2}mv^2 = KE_T$$

The two equations are the equal.
You can use either equation to describe the kinetic energy of a point particle moving in a *circle*.

You can do the same thing with Angular Momentum of a Point Particle versus a Rigid Object with Shape:
$$L_{point\ particle} = rmv\sin\theta \ \& \ L_{\substack{Rigid\ Object\\with\ Shape}} = I\omega$$

$$L_{point\ particle} = rmv\sin\theta = Rm\left(R\omega\right)\sin\left(90\right) = \left(mR^2\right)\omega = I\omega = L_{\substack{Rigid\ Object\\with\ Shape}}$$

The two equations are the equal.
You can use either equation to describe the angular momentum of a point particle moving in a *circle*.

What if the point particle is moving in an *ellipse*? (Like a satellite)
- Nothing changes in the derivation for kinetic energy, which is a scalar. Therefore, you can still use either the translational or rotational equation for kinetic energy.
- Something does change for angular momentum, which is a vector. The angle between the r vector and velocity is no longer 90 degrees and our derivation is no longer valid.
  - We **cannot** use the equation $L_{\substack{Rigid\ Object\\with\ Shape}} = I\omega$ for a point particle moving in an ellipse, we have to use $L_{point\ particle} = rmv\sin\theta$.

Example: A satellite is in an elliptical orbit. Is its _____ momentum conserved? (a) linear, (b) angular.

(a) Linear momentum is conserved when the net force acting on the system equals zero:

$$\sum \vec{F}_{\substack{exrternal \\ on\ system}} = \frac{\Delta \vec{p}}{\Delta t} = 0 \Rightarrow \sum \vec{p}_i = \sum \vec{p}_f$$

The system in this case is just the satellite. So, is the net force acting on an object in an elliptical orbit equal to zero? Let's draw a free body diagram of the forces acting on the satellite.

FBD: Fg toward center of the primary object which is the object the satellite is orbiting around.

There is only one force acting on the satellite, the force of gravitational attraction caused by the primary, therefore the net force acting on the satellite cannot be zero. So, linear momentum *is not conserved* for an object in an elliptical orbit.

$$\sum \vec{F}_{\substack{exrternal \\ on\ Satellite}} = \frac{\Delta \vec{p}}{\Delta t} \neq 0 \Rightarrow \sum \vec{p}_i \neq \sum \vec{p}_f$$

(b) Angular momentum is conserved when the net torque acting on the system equals zero:

$$\sum \vec{\tau}_{\substack{external \\ on\ system}} = \frac{\Delta \vec{L}_{system}}{\Delta t} = 0 \Rightarrow \sum \vec{L}_i = \sum \vec{L}_f$$

Again, the system is just the satellite. So, is the net torque acting on an object in an elliptical orbit equal to zero? We have already drawn the free body diagram, however, before we can answer the question it is important to remember we have to identify an axis of rotation when working with net torque and angular momentum. Let's define the axis of rotation for the satellite as the center of the primary. Because the only force acting on the satellite is the force of gravitational attraction which points directly toward the center of mass of the primary, the torque caused by this force equals zero, and therefore angular momentum of a satellite in elliptical orbit *is conserved* about the center of mass of the primary.

$$\sum \vec{\tau}_{\substack{external \\ on\ Satellite \\ with\ AoR\ at\ Primary}} = \frac{\Delta \vec{L}_{system}}{\Delta t} = 0 \Rightarrow \sum \vec{L}_i = \sum \vec{L}_f$$

I will point out that, if we assume the orbit is circular, the answers to both of our questions are the same. I simply identified the orbit as elliptical because most orbits are nearly elliptical and very few orbits are nearly circular.

Flipping Physics Lecture Notes:

Are Linear and Angular Momentum Conserved during this Collision?

Example: As shown, a point particle strikes a wooden board mounted such that it is free to rotate about its center. During this collision, is _____ momentum of the point particle and wooden board system conserved? (a) Linear, (b) Angular.

Pictures do not do the example justice.
Please watch the video.
https://www.flippingphysics.com/collision-angular-momentum-conservation.html
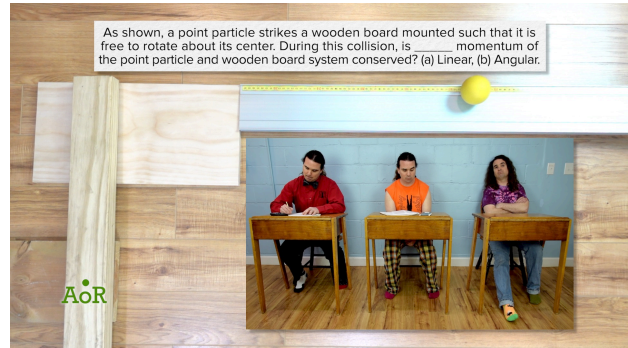
Let's start with linear momentum. As the point particle moves toward the wooden board, it is moving to the left and therefore has linear momentum which is to the left. After the collision, the point particle is moving to the right and therefore has linear momentum which is to the right. So, clearly the linear momentum of the point particle changes and is not conserved. However, we are talking about the linear momentum of the point particle and wooden board system. So, what about the linear momentum of the wooden board?

Before the collision the wooden board is not moving, so it has zero linear momentum. After the collision the wooden board is rotating, however, its center of mass is not moving, so it still has zero linear momentum. Therefore, the wooden board has zero change in its linear momentum.

Summing the linear momenta of the wooden board and the point particle, there must have been a change in linear momentum of the system because the linear momentum of the point particle changed, and the linear momentum of the wooden board did not. Therefore, linear momentum of the point particle and wooden board system is **not** conserved.

Looking at the equations, linear momentum is conserved when the net force on the system equals zero:

$$\sum \vec{F}_{system} = \frac{\Delta \vec{p}}{\Delta t} = 0 \Rightarrow \Delta \vec{p} = 0 = \vec{p}_f - \vec{p}_i \Rightarrow \vec{p}_{i\,system} = \vec{p}_{f\,system} \text{ or } \sum \vec{p}_i = \sum \vec{p}_f$$

Therefore, the net force on the system in this example must *not* equal zero. When the point particle collides with the board, the point particle applies a force on the board and, according to Newton's Third Law, the board applies an equal but opposite force on the point particle. Those two forces are internal to the system and cancel one another out. So what is another force which is causing the net force on the system to *not* equal zero?

Because the board is attached to a Lazy Susan which allows the board to rotate but not change locations, while the point particle strikes the board, the pivot point applies an external force on the board which prevents the board from entering into translational motion. That means the net external force acting on the board and point particle system during the collision does not equal zero, and linear momentum of the system is not conserved. Please realize this force from the pivot point only acts on the board and point particle system while the point particle is in contact with the board which is only a very short time interval. Also, I will point out, however, that, if the board were not held in place at the pivot point and were instead on a frictionless surface, there would be no net external force acting on the board and point particle system, and linear momentum of the system would be conserved. After that collision, the board would enter in to both rotational and translational motion.
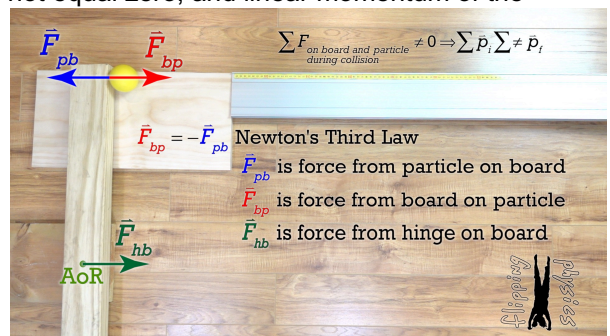
What about angular momentum? Is the angular momentum of the point particle and wooden board system conserved? Going back to looking at the directions of the angular momenta. As the point particle is moving toward the board, it has angular momentum which is, according to the right-hand rule, out of the screen. At this point the board is not moving and has zero angular momentum. So, before the collision, the system has angular momentum which is out of the screen.
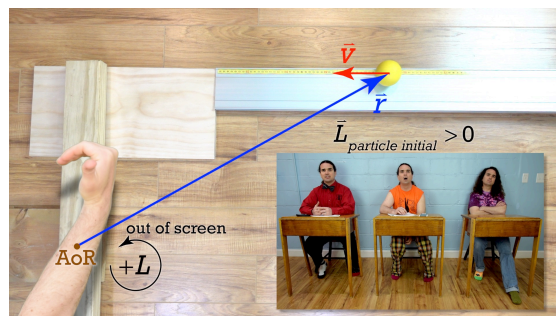
After the collision the point particle is moving in the opposite direction and has, according to the right-hand rule, angular momentum which is into the screen. The board now has an angular momentum which is out of the screen. Therefore, the angular momentum before the collision is out of the screen and after the collision is the summation of the point particle's into the screen and wooden board's out of screen angular momenta. In other words, this approach is inconclusive. We cannot tell right now if the angular momentum of the point particle and wooden board system is conserved during this collision. We are going to have to rely on the equations.
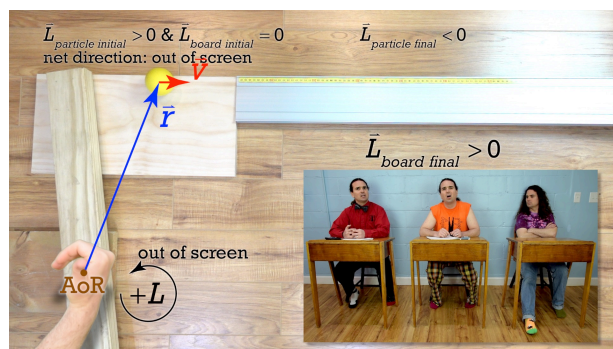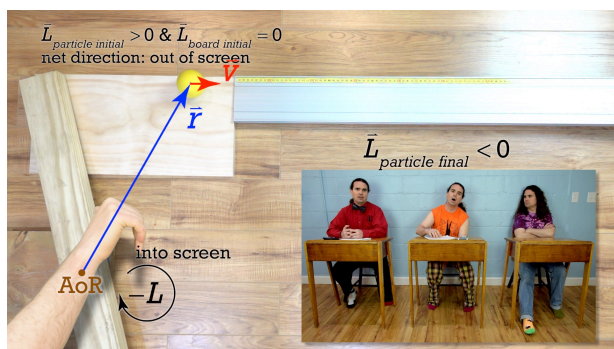
Angular momentum is conserved when the net torque on the system equals zero:

$$\sum \vec{\tau}_{system} = \frac{\Delta \vec{L}}{\Delta t} = 0 \Rightarrow \Delta \vec{L} = 0 = \vec{L}_f - \vec{L}_i \Rightarrow \vec{L}_{i\ system} = \vec{L}_{f\ system} \text{ or } \sum \vec{L}_i = \sum \vec{L}_f$$

We already have discussed all the forces acting on the system, so let's see how those cause torques on the system. Let's sum the torques about the axis of rotation of the wooden board.

The two torques caused by the Newton's Third Law Force Pair of forces (the particle on the board and the board on the particle) have the same r vector and their two angels are supplementary, which means they have the same value for sine. Therefore, those two torques are equal and opposite and cancel one another out.

What about the centripetal forces internal to the wooden board which cause the board to rotate? By definition, those centripetal forces are directed in toward the axis of rotation of the board. Therefore, the r vector and force vector are opposite in direction and the angle between those two is 180 degrees. The sine of 180 degrees equals zero. So the centripetal forces internal to the wooden board cause zero torque on the board. The net torque on the system then equals zero and the angular momentum of the point particle and wooden board system *is conserved* about the axis of rotation of the wooden board!

Flipping Physics Lecture Notes:

Point Particle with Rigid Object Collision
Conservation of Angular Momentum Demonstration
and Problem

Example: As shown, a 5.3 g dart moving horizontally at 16.9 m/s collides with and sticks to a stationary R̲otational I̲nertia D̲emonstrator a distance of 31.7 cm from the axis of rotation of the RID. What is the final angular velocity of the RID? I_{RID} = 0.0237 kg·m²
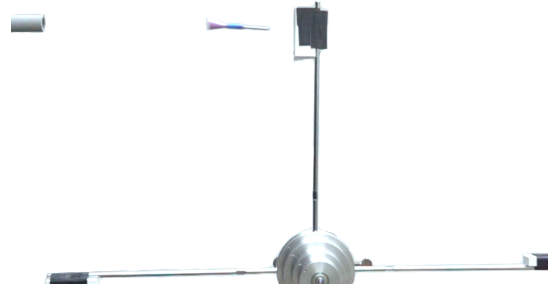
Knowns: $m = 0.0053kg$; $v_{di} = 16.9\dfrac{m}{s}$; $y = 0.317m$; $\omega_{RIDi} = 0$; $I_{RID} = 0.0237kg \cdot m^2$; $\omega_{RIDf} = ?$

We have already determined that, about the axis of rotation of the rigid object with shape, angular momentum is conserved in a situation where a point particle (the dart) collides with a rigid object with shape (the RID). https://www.flippingphysics.com/collision-angular-momentum-conservation.html

$$\sum \vec{\tau}_{\substack{\text{dart and RID} \\ \text{AoR @ center of RID}}} = \frac{\Delta \vec{L}}{\Delta t} = 0 \Rightarrow \sum \vec{L}_i = \sum \vec{L}_f \Rightarrow L_{di} + L_{RIDi} = L_{df} + L_{RIDf}$$

- We currently have two equations for angular momenta of objects:
$$L_{\text{point partilce}} = rmv\sin\theta \ \& \ L_{\substack{\text{Rigid Object} \\ \text{with Shape}}} = I\omega$$
- For the dart before the collision, we use the equation for the angular momentum of a point particle: $L_{di} = r_{di}m_d v_{di}\sin\theta_{di}$
  - We are treating the dart as a point particle, so you will see me refer to it as both a "dart" and a "point particle", depending on the situation.
- Before the collision the RID is not moving, so it does not have any angular momentum.
- For the dart after the collision, we have shown that we can use either of the two equations for the angular momentum of the point particle because it is moving in a circle. Let's use: $L_{df} = I_{df}\omega_{df}$
https://www.flippingphysics.com/point-particle-kinetic-energy-and-angular-momentum-equations.html
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- For the RID after the collision, we have to use the equation for the angular momentum of a rigid object with shape: $L_{RIDf} = I_{RID}\omega_{RIDf}$

$$\Rightarrow r_{di}m_d v_{di}\sin\theta_{di} = I_{df}\omega_{df} + I_{RID}\omega_{RIDf}$$

- We have already shown that, while r_{id} and θ_{id} change as the dart moves, y, which equals $r_{id}\sin\theta_{id}$ does not change:
$$\sin\theta_{di} = \frac{O}{H} = \frac{y}{r_{di}} \Rightarrow y = r_{di}\sin\theta_{di}$$
https://www.flippingphysics.com/collision-angular-momentum-conservation.html
- The equation for rotational inertia of a point particle is: $I_{df} = m_d r_{df}^2$

$$\Rightarrow ym_d v_{di} = m_d r_{df}^2 \omega_{df} + I_{RID}\omega_{RIDf}$$

- The dart stays the distance "y" from the axis of rotation after it sticks to the RID: $r_{df} = y$

- Both the dart and the RID are rotating together after the collision: $\omega_{df} = \omega_{RIDf} = \omega_f$

$$\Rightarrow ym_d v_{di} = m_d y^2 \omega_f + I_{RID}\omega_f = \left(m_d y^2 + I_{RID}\right)\omega_f \Rightarrow \omega_f = \frac{ym_d v_{di}}{m_d y^2 + I_{RID}}$$

$$\Rightarrow \omega_f = \frac{(0.317)(0.0053)(16.9)}{(0.0053)(0.317^2) + 0.0237} = 1.1717 \approx \boxed{1.2 \frac{rad}{s}}$$

This is our predicted final angular velocity of the system.

Measured final velocity of the system:

$$Knowns: \Delta\theta = 5.0° \times \frac{2\pi rad}{360°} = \frac{\pi}{36} rad; \Delta t = 69 frames \times \frac{1sec}{960 frames} = 0.071875 \sec$$

$$\omega_{measured} = \frac{\Delta\theta}{\Delta t} = \frac{\frac{\pi}{36} rad}{0.071875 \sec} = 1.21414 \frac{rad}{s}$$

$$Percent\ Difference = \frac{1.21414 - 1.17171}{\left(1.21414 + 1.17171\right)\Big/ 2} \times 100 = 3.5566 \approx 3.6\%$$

If you were curious where the rotational inertia of the rotational inertia demonstrator came from, it is the addition of the rotational inertia of it without masses and the rotational inertia of the four 26.1 gram masses which are each 31.0 cm from the axis of rotation. We measured the rotational inertia of the rotational inertia demonstrator in this video: https://www.flippingphysics.com/thin-rod-rotational-inertia.html

$$I_{RID} = I_{RID(nomasses)} + 4I_{one\ mass} = I_{RID(nomasses)} + 4mr^2 = 0.0137 + (4)(0.0260)(0.310)^2 = 0.023733 \approx 0.0237 kg \cdot m^2$$
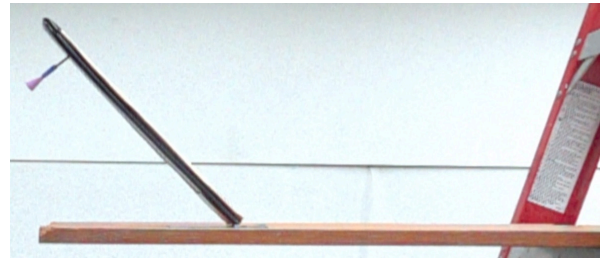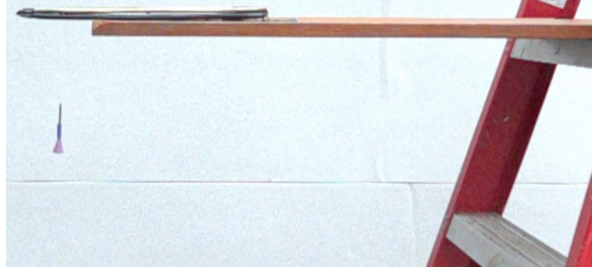
Example: As shown, a 5.3 g dart is moving vertically at 16.5 m/s just before it collides with and sticks to a 33.9 cm long, thin piece of cardboard. If the dart hits the 71.8 g piece of cardboard 28.7 cm from its fixed

$$I_{end} = \frac{1}{3}mL^2$$

end, to what maximum angle does the cardboard rise?

Knowns: $m_d = 0.0053kg; v_{di} = 16.5\frac{m}{s}; m_c = 0.0718kg; x = 0.287m; L_c = 0.339m; \theta_f = ?$

$$I_{end} = \frac{1}{3}mL^2 = I_c = \frac{1}{3}(0.0718)(0.339)^2 = 0.0027504 kg \cdot m^2$$

We have already determined that, about the axis of rotation of the rigid object with shape, angular momentum is conserved in a situation where a point particle collides with a rigid object with shape.
https://www.flippingphysics.com/collision-angular-momentum-conservation.html

However, the collision is just part 1 of the problem, part 2 of the problem is conservation of mechanical energy as the cardboard and dart rise together. We know mechanical energy is conserved while the cardboard rotates because there is no energy added to or removed from the system via work done by a force of friction or force applied.

$$\sum \vec{\tau}_{\substack{\text{dart and cardboard} \\ \text{AoR @ fixed end of cardboard}}} = \frac{\Delta \vec{L}}{\Delta t} = 0 \Rightarrow \sum \vec{L}_{1i} = \sum \vec{L}_{1f} \Rightarrow L_{d1i} + L_{c1i} = L_{d1f} + L_{c1f} \ \& \ ME_{2i} = ME_{2f}$$

- We currently have two equations for angular momenta of objects:

  $$L_{\text{point partilce}} = rmv\sin\theta \ \& \ L_{\substack{\text{Rigid Object} \\ \text{with Shape}}} = I\omega$$

- For the dart before the collision, we use the equation for the angular momentum of a point
  particle: $L_{d1i} = r_{d1i} m_d v_{d1i} \sin\theta_{d1i}$
  - We are treating the dart as a point particle, so you will see me refer to it as both a "dart"
    and a "point particle", depending on the situation.
- Before the collision the cardboard is not moving, so it does not have any angular momentum.
- For the dart after the collision, we have shown that we can use either of those two equations for
  the angular momentum of the point particle because it is moving in a circle. Let's use:

  $$L_{d1f} = I_{d1f}\omega_{d1f}$$
  https://www.flippingphysics.com/point-particle-kinetic-energy-and-angular-momentum-equations.html

- For the cardboard after the collision, we have to use the equation for the angular momentum of a rigid object with shape: $L_{c1f} = I_c \omega_{c1f}$

$$\Rightarrow r_{d1i} m_d v_{d1i} \sin \theta_{d1i} = I_{d1f} \omega_{d1f} + I_c \omega_{c1f}$$

- We have already shown that, while r_{id} and θ_{id} change as the dart moves, x, which equals

$$\sin \theta_{d1i} = \frac{O}{H} = \frac{x}{r_{d1i}} \Rightarrow x = r_{d1i} \sin \theta_{d1i}$$

$r_{d1i} \sin \theta_{d1i}$ does not change:
https://www.flippingphysics.com/angular-momentum-triangle.html

- The equation for rotational inertia of a point particle is: $I_{d1f} = m_d r_{d1f}^2$

$$\Rightarrow x m_d v_{d1i} = m_d r_{d1f}^2 \omega_{d1f} + I_c \omega_{c1f}$$

- The dart stays the distance "x" from the axis of rotation after it sticks to the cardboard: $r_{d1f} = x$
- Both the dart and the cardboard are rotating together after the collision and the final angular velocity of part 1 is the same as the initial angular velocity for part 2: $\omega_{d1f} = \omega_{c1f} = \omega_{1f} = \omega_{2i}$

$$\Rightarrow x m_d v_{d1i} = m_d x^2 \omega_{1f} + I_c \omega_{1f} = \left(m_d x^2 + I_c \right) \omega_{1f} \Rightarrow \omega_{1f} = \frac{x m_d v_{d1i}}{m_d x^2 + I_c} = \omega_{2i}$$

$$\Rightarrow \omega_{1f} = \frac{(0.287)(0.0053)(16.5)}{(0.0053)(0.287)^2 + 0.0027504} = 7.87517 \frac{rad}{s} = \omega_{2i}$$

- And now for part 2 we use conservation of mechanical energy: $ME_{2i} = ME_{2f}$
  - The initial point is right after the dart collides with the cardboard and before the cardboard starts to rise. The final point is at the maximum angle the cardboard rises to. Lets set the horiztonal zero line at the initial height of the dart and cardboard.
- There are no springs, so no elastic potential energy in the problem.
- Initially the dart and cardboard are at the horizontal zero line, so they do not have any gravitational potential energy initial.
- The only type of initial mechanical energy is kinetic energy and because the system is rotating, it is the combined rotational kinetic energy initial of both the dart and the cardboard.
- At the final point, the dart and cardboard are not moving, so they have no final kinetic energy.
- At the final point, the dart and cardboard are above the zero line, so they have final gravitational potential energy.

$$\Rightarrow \frac{1}{2} I_{system} \omega_{2i}^2 = m_d g h_{d2f} + m_c g h_{c2f}$$

- The rotational inertia of the system is: $I_{system} = I_d + I_c = m_d r_{d2f}^2 + I_c = m_d x^2 + I_c$

  - The distance from the dart to the axis of rotation is still x: $r_{d2f} = x$

$$\Rightarrow \frac{1}{2} \left(m_d x^2 + I_c \right) \omega_{2i}^2 = m_d g h_{d2f} + m_c g h_{c2f}$$

- Notice the only two variables we do not know in this equation are the two height finals.
  - In order to reduce this equation to one unknown, we need to draw triangles and use trigonometry to find relationships between the angle, θ, between the cardboard and the horizontal, and both height finals.
  - The height for part 2 final of the cardboard is to the center mass of the cardboard. We will consider the cardboard to be of uniform density, so the center of mass is in the middle of the cardboard.
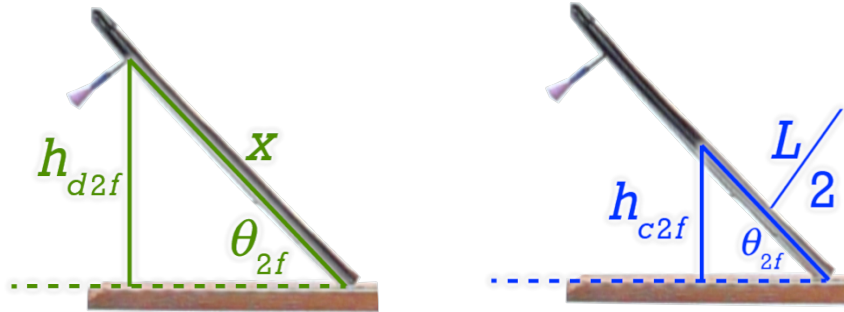
$$\sin\theta_{2f} = \frac{O}{H} = \frac{h_{d2f}}{x} \Rightarrow h_{d2f} = x\sin\theta_{2f} \ \& \ \sin\theta_{2f} = \frac{O}{H} = \frac{h_{c2f}}{L/2} \Rightarrow h_{c2f} = \frac{L}{2}\sin\theta_{2f}$$

$$\Rightarrow \frac{1}{2}\left(m_d x^2 + I_c\right)\omega_{2i}^{\ 2} = m_d gx\sin\theta_{2f} + m_c g\frac{L}{2}\sin\theta_{2f} = \left(m_d gx + m_c g\frac{L}{2}\right)\sin\theta_{2f}$$

$$\Rightarrow \sin\theta_{2f} = \frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\omega_{2i}^{\ 2}}{m_d gx + m_c g\frac{L}{2}} \Rightarrow \theta_{2f} = \sin^{-1}\left(\frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\omega_{2i}^{\ 2}}{m_d gx + m_c g\frac{L}{2}}\right)$$

$$\Rightarrow \theta_{2f} = \sin^{-1}\left(\frac{\left(\frac{1}{2}\right)\left[(0.0053)(0.287)^2 + 0.0027504\right](7.87517)^2}{(0.0053)(9.81)(0.287) + (0.0718)(9.81)\left(\frac{0.339}{2}\right)}\right) = 48.101° \approx \boxed{48°}$$

This is our predicted final angle of the cardboard.

$$\text{Measured } \theta_{2f} = 46°$$

$$\text{Percent Difference} = \frac{48.101 - 46}{\left(48.101 + 46\right)/2} \times 100 = 4.46603 \approx 4.5\%$$

Alternative solution with not solving for the initial angular velocity for part 2:

$$\omega_{2i} = \frac{x m_d V_{d1i}}{m_d x^2 + I_c} \ \& \ \theta_{2f} = \sin^{-1}\left(\frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\omega_{2i}^2}{m_d g x + m_c g \frac{L}{2}} \right) = \sin^{-1}\left(\frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\left(\frac{x m_d V_{d1i}}{m_d x^2 + I_c}\right)^2}{m_d g x + m_c g \frac{L}{2}} \right)$$

$$\Rightarrow \theta_{2f} = \sin^{-1}\left(\frac{\frac{1}{2}\left(m_d x^2 + I_c\right)\frac{\left(x m_d V_{d1i}\right)^2}{\left(m_d x^2 + I_c\right)^2}}{m_d g x + m_c g \frac{L}{2}} \right) = \sin^{-1}\left(\frac{\frac{\left(x m_d V_{d1i}\right)^2}{m_d x^2 + I_c}}{2 m_d g x + m_c g L} \right)$$

$$\Rightarrow \theta_{2f} = \sin^{-1}\left(\frac{\left(x m_d V_{d1i}\right)^2}{\left(m_d x^2 + I_c\right)\left(2 m_d g x + m_c g L\right)} \right)$$