Flipping Physics Lecture Notes:

Newton's Universal Law of Gravitation Introduction

Remember: Force of Gravity and Weight mean the same thing. The equation we already have for this is:

- $F_g = mg$

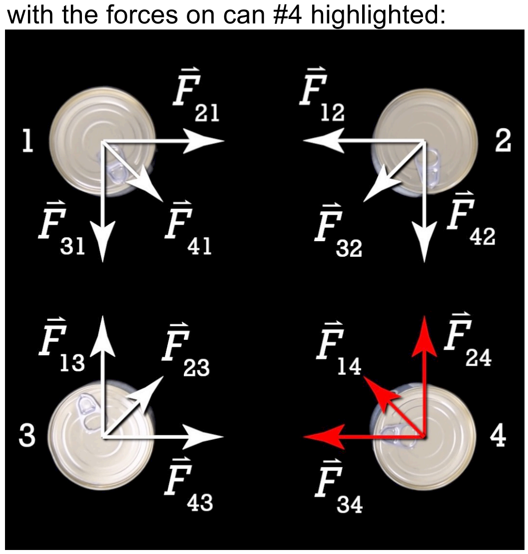
However, this is missing a subscript that has been assumed up until this point:

- $F_g = m_o g$: m_o means the *mass of the object*. This equation is for the Force of Gravity that exists between and object and a planet. Usually for us the planet is the Earth. $g_{Earth} = 9.81\frac{m}{s^2}$

Truth is that a force of gravitational attraction exists between any two objects. The equation to determine this force of gravitational attraction is Newton's Universal Law of Gravitation:

- $\boxed{F_g = \frac{Gm_1m_2}{r^2}}$
  - G is the Universal Gravitational Constant. $\boxed{G = 6.67\times10^{-11}\frac{N\cdot m^2}{kg^2}}$
  - I call this The Big G Equation!
  - m_1 and m_2 are the masses of the two objects.
  - r is not defined as the radius.
    - r is the distance between the centers of mass of the two objects.
    - r sometimes is the radius.
  - Equation was established by Sir Isaac Newton in 1687.
  - Not until 1796 was the Universal Gravitational Constant first measured by British Scientist Henry Cavendish. He used a large torsion balance to measure G.

The cans of dog food example with the forces on can #4 highlighted:



An interesting point: According to The University of Oxford Department of Physics,[*] "Cavendish used the balance in 1798 and measured the mean density of the earth at 5.48g/cm3. This implied that G was 6.754x10-11m3s-2kg-1 although Cavendish did not derive it." Therefore Cavendish actually never calculated G. I apologize for this oversight! Thank you to Dan Burns @kilroi22 for pointing out this error.

[*] http://www.physics.ox.ac.uk/history.asp?page=BigGHis

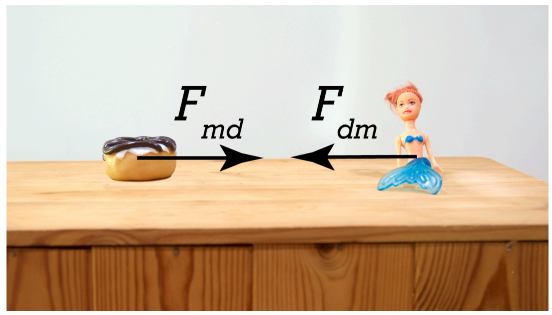
Flipping Physics Lecture Notes:

How Much is a Mermaid Attracted to a Doughnut?
(Newton's Universal Law of Gravitation)

Example: A 17 g doughnut is sitting 21 cm from a 14 g mermaid. Determine the force of gravitational attraction between the two.

Knowns: $m_d = 17g\left(\frac{1kg}{1000g}\right) = 0.017kg;\ m_m = 14g\left(\frac{1kg}{1000g}\right) = 0.014kg;$

$r = 21cm\left(\frac{1m}{100cm}\right) = 0.21m;\ F_g = ?$



$$F_g = \frac{Gm_1m_2}{r^2} = \frac{Gm_dm_m}{r^2} = \frac{(6.67\times10^{-11})(0.017)(0.014)}{0.21^2} = 3.59968\times10^{-13} \approx \boxed{3.6\times10^{-13}N}$$

In an otherwise empty universe, the doughnut and mermaid meet at their two-object center of mass:

$$x_{cm} = \frac{m_dx_d + m_mx_m}{m_d + m_m} = \frac{(17)(0)+(14)(21)}{17+14} \Rightarrow x_{cm} = 9.4839 \approx 9.5\text{cm}$$

Flipping Physics Lecture Notes:

The Force of Gravitational Attraction between the Earth and the Moon

Example: According to NASA, the mass of the Earth is 5.97 x 10^{24} kg, the mass of the Moon is 7.3 x 10^{22} kg, and the mean distance between the Earth and the Moon is 3.84 x 10^{8} m.♥ What is the force of gravitational attraction between the Earth and the Moon?

Knowns: $m_{Earth} = 5.97\times10^{24}kg;\ m_{Moon} = 7.3\times10^{22}kg;\ R_{E-M} = 3.84\times10^{8}m;\ F_g = ?$

$$F_g = \frac{Gm_1m_2}{r^2} = \frac{Gm_Em_M}{R_{E-M}{}^2} = \frac{\left(6.67\times10^{-11}\right)\left(5.97\times10^{24}\right)\left(7.3\times10^{22}\right)}{\left(3.84\times10^{8}\right)^2}$$

$$\Rightarrow F_g = 1.97134\times10^{20} \approx \boxed{2.0\times10^{20}N}$$

There are a number of things wrong with the following calculations:

$$\sum F_{moon} = F_{EM} = m_m a_m \Rightarrow a_m = \frac{F_{EM}}{m_m} = \frac{1.97134\times10^{20}}{7.3\times10^{22}} = 0.0027005\frac{m}{s^2} \approx 0.0027\frac{m}{s^2}$$

$$\Delta x = v_i\Delta t + \frac{1}{2}a\Delta t^2 \Rightarrow 3.84\times10^{8} = \left(0\right)\Delta t + \frac{1}{2}\left(0.0027005\right)\Delta t^2$$

$$\Rightarrow \Delta t = \sqrt{\frac{\left(2\right)\left(3.84\times10^{8}\right)}{0.0027005}} = 533288\sec\left(\frac{1hr}{3600\sec}\right)\left(\frac{1day}{24hr}\right) = 6.17231 \approx 6.2days$$

It is important you understand what is wrong with them though…

♥ https://nssdc.gsfc.nasa.gov/planetary/factsheet/

Flipping Physics Lecture Notes:

Deriving the Acceleration due to Gravity on any Planet and specifically Mt. Everest.

We have two equations for the force of gravity acting on an object. When an object is near the surface of a planet, we can set the two equal to one another:

$$F_g = m_o g = \frac{Gm_1 m_2}{r^2} \Rightarrow m_o g = \frac{Gm_o m_p}{\left(R_p + altitude\right)^2} \Rightarrow g_{planet} = \frac{Gm_p}{\left(R_p + altitude\right)^2}$$

We just solved for the acceleration due to gravity on planet Earth (or any planet for that matter). Notice the acceleration due to gravity is not actually constant. ☺

Let's determine the acceleration due to gravity on Mt. Everest. In order to do so, we need some known values:
The mass of the Earth is 5.9723×10^{24} kg.♥
The radius of the Earth at Mt. Everest is 6.3735×10^6 m.
The altitude of Mt. Everest is 8,848 m. The top of Mt. Everest is 8,848 meters above sea level.

$$\Rightarrow g_{Mt.Everest} = \frac{\left(6.67\times10^{-11}\right)\left(5.9723\times10^{24}\right)}{\left(6.3735\times10^{6}+8848\right)^2} = 9.77468 \approx \boxed{9.77\frac{m}{s^2}}$$

So yes, the acceleration due to gravity is not constant on the surface of planet Earth, however, it is pretty darn close.

Please note: The Earth is not a perfect sphere; it is an oblate spheroid. Its equatorial radius is larger than it's polar radius. The larger equatorial radius is caused by the rotation of the planet and it's own inertia.

- The Equatorial radius of the Earth is 6.378×10^6 m.
- The radius of the Earth at the poles is 6.357×10^6 m.
- The average radius of the Earth is 6.371×10^6 m.
- In order to determine the radius of the Earth at Mt. Everest, we need to know the latitude of Mt. Everest. It is 27.986065° North.♣
- We can use the latitude of Mt. Everest to determine the radius of the Earth at Mt. Everest. It is 6.3735×10^6 m.♦

World record holder Javier Sotomayor, 1993, Salamanca, Spain:

$\Delta y = 2.45m;\ a_y = -9.80\frac{m}{s^2};\ v_{fy} = 0$ at top of path

$$v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y = 0^2 \Rightarrow v_{iy}^2 = -\left(2\right)\left(-9.80\right)\left(2.45\right) \Rightarrow v_{iy} = \sqrt{-\left(2\right)\left(-9.80\right)\left(2.45\right)} = 6.92965\frac{m}{s}$$

If he jumped instead on Mt. Everest: $v_{iy} = 6.92965\frac{m}{s};\ a_y = -9.77\frac{m}{s^2};\ v_{fy} = 0$ at top of path

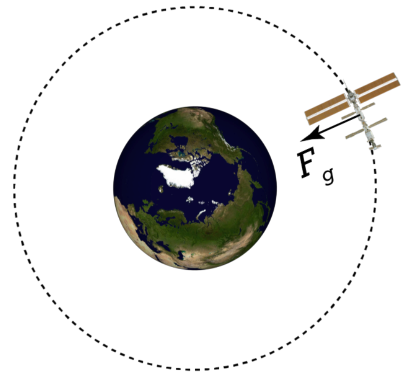
$$v_{fy}^2 = v_{iy}^2 + 2a_y\Delta y = 0^2 \Rightarrow v_{iy}^2 = -2a_y\Delta y \Rightarrow \Delta y = \frac{v_{iy}^2}{-2a_y} = \frac{\left(6.92965\right)^2}{-\left(2\right)\left(-9.77\right)} = 2.45752 \approx 2.46\text{m}$$

He would have been able to jump about 1 mm higher! But what about the oxygen tanks, snow, etc?

♥ https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html
♣ http://www.latlong.net/place/mount-everest-nepal-14.html
♦ https://rechneronline.de/earth-radius/

Flipping Physics Lecture Notes:

Altitude of Geosynchronous Orbit
(a.k.a. Geostationary Orbit)



Example: What is the altitude of a satellite in geosynchronous orbit?

Knowns: $m_{Earth} = 5.9723\ x\ 10^{24}\,kg;\ R_{Earth-Equatorial} = 6.378\ x\ 10^{6}\,m$ ♥

Geostationary orbit is where the object orbiting the Earth is always above the same spot on the Earth. Note: In order for this to work, the orbiting object must be above the Earth's equator.♦

$$\sum F_{in} = F_g = ma_c \Rightarrow \frac{Gm_1m_2}{r^2} = mr\omega^2$$

Please note the error in the video: Geosynchronous orbit and geostationary orbit are not the same. Geostationary orbit is a special case of geosynchronous orbit. A geosynchronous orbit simply has the same 24 hour period as the Earth, however, it is inclined relative to the equator and traces out an ellipse in the sky as seen from the Earth. Sorry they are incorrectly identified as the same in the video. Thank you to Dan Burns @kilroi22 and Christopher Becke @BeckePhysics for the correction!

Because the satellite is always above the same spot on the Earth, the satellite has the same angular velocity as the earth. In other words, the satellite goes through one revolution every 24 hours.

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi rad}{24hr} \times \frac{1hr}{3600s} = 7.27221 \times 10^{-5}\,\frac{rad}{s}$$

We need the angular velocity in terms of seconds. This is because G is in $\frac{N \cdot m^2}{s^2}$.

$r = R_{Earth} + Altitude$; Distance from center of mass of Earth to center of mass of satellite.

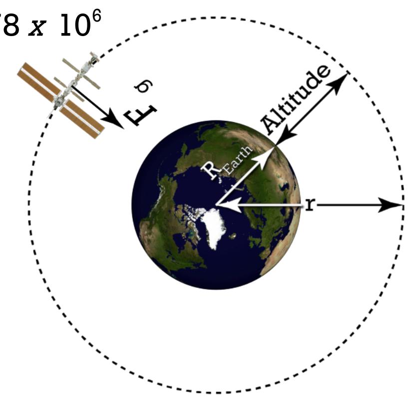
Note: r is the same on both sides of the equation.

$$\Rightarrow \frac{Gm_sm_E}{r^2} = m_sr\omega^2 \Rightarrow \frac{Gm_E}{r^2} = r\omega^2 \Rightarrow \frac{Gm_E}{r^3} = \omega^2 \Rightarrow r^3 = \frac{Gm_E}{\omega^2} \Rightarrow r = \sqrt[3]{\frac{Gm_E}{\omega^2}}$$

$$r = \sqrt[3]{\frac{\left(6.67 \times 10^{-11}\right)\left(5.9723\ x\ 10^{24}\right)}{\left(7.27221 \times 10^{-5}\right)^2}} = 4.22323 \times 10^{7}\,m$$

$$r = R_{Earth} + Altitude \Rightarrow Altitude = r - R_{Earth} = 4.22323 \times 10^{7} - 6.378\ x\ 10^{6}$$

$$\Rightarrow r = 3.58543 \times 10^{7} \approx \boxed{3.59 \times 10^{7}\,m}$$



Note: NASA lists the altitude of geosynchronous orbit as 35,900 km.♣

$$35{,}900km \times \frac{1000m}{1km} = 3.59 \times 10^{7}\,m;$$ Yep, we got it right!

♥ https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html

♦ Just so you know, this means all satellite dishes on planet Earth that communicate with a geostationary satellite will point towards a location above the equator.

♣ https://www.nasa.gov/multimedia/imagegallery/image_feature_388.html

Flipping Physics Lecture Notes:

Dropping a Bucket of Water Demonstration

Water flows out of a bucket with two holes in it which filled with water because he force of gravity pulls the water down, but you are holding the bucket up. The bucket pushes up on the water to hold the water up, but where the two holes are, the bucket does not hold the water. So the force of gravity pulls the water out of the bucket.

Water stops flowing out of the bucket after you drop it because when you drop the bucket, the force of gravity is still acting on the bucket and the water; however, you are no longer holding the bucket up. So the only force acting on the bucket and water is the force of gravity. The whole thing enters free fall and accelerates downward at 9.81 meters per second squared. In other words, the bucket no longer gets in the way of the water and the water and bucket both fall straight down.

Another way of looking at it is to think about the bottom of the bucket relative to the water. The bottom of the bucket is always accelerating downward at the same rate as the water, so the bottom of the bucket is basically always getting out of the way of the water because they are both in free fall. Therefore the water is never able to catch up to the holes and the water falls straight down.

A reason the water does not immediately stop flowing out of the holes when the bucket is dropped at takes about half a second to stop is before Mr. P. drops the bucket, some of the water inside the bucket is flowing toward the holes. When he drops the bucket, the water stops being pushed out of the holes by the force of gravity, however, the inertia of the water which was already flowing towards the holes keeps the water flowing towards and out of the holes. So the inertia of the water, or the tendency of the water to try to maintain a constant velocity, causes it to take about half a second for water to stop flowing out of the holes of the bucket.

Flipping Physics Lecture Notes:

Apparent Weightlessness

In order for an object to have zero weight, the force of gravity acting on the object would need to be zero.

$$F_g = m_o g = m_o \left(0\right) = 0$$

In order for that to be true, the acceleration due to gravity would need to be zero.

$$F_g = m_o g = \frac{G m_o m_E}{r^2} \Rightarrow g = \frac{G m_E}{r^2}$$

In order for the acceleration due to gravity to be zero, the object would need to be infinitely far away from all other objects ($r \approx \infty$), which is impossible. Okay, so this is the simplest way to get something to be weightless, however, there are others. However, I am more concerned about the fact that Astronauts in space are often referred to as weightless, however, because none of those cases you just mentioned apply to Astronauts in space, astronauts in space are not weightless.

Solving for the acceleration due to gravity on the International Space Station.

Knowns: $Altitude_{ISS\ average} = 4.2 \times 10^5\, m;\ M_{Earth} = 5.97 \times 10^{24}\, kg;\ R_{Earth} = 6.371 \times 10^6\, m$ ♠

Note: The altitude of the International Space Station varies from 304 to 528 kilometers. We have taken an average of 420 km.♥

$$g_{ISS\ average} = \frac{G m_E}{\left(R_{Earth} + Altitude_{ISS}\right)^2} = \frac{\left(6.67 \times 10^{-11}\right)\left(5.97 \times 10^{24}\right)}{\left(6.371 \times 10^6 + 4.16 \times 10^5\right)^2} = 8.63441 \approx 8.6 \frac{m}{s^2}$$

$$\frac{g_{ISS\ average}}{g_{Earth}} \times 100 = \frac{8.63441}{9.81} \times 100 = 88.0164 \approx 88\%$$

Objects in orbit are not weightless, however, they appear to be weightless because everything is falling at the same rate, hence, “apparent weightlessness”.

♠ https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html
♥ https://www.nasa.gov/centers/kennedy/about/information/shuttle_faq.html#14

Flipping Physics Lecture Notes:

Number of g's or g-Forces

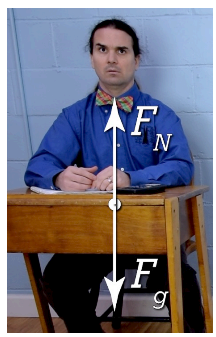
I have two different equations that I use to determine the number of g’s, or what are also called g-Forces, which act on an object, they are:

- $vertical\ number\ of\ g's = \dfrac{F_N}{F_{g\ Earth}}$
- $horizontal\ number\ of\ g's = \dfrac{a_x}{g_{Earth}}$

Remember: $g_{Earth} = +9.81\dfrac{m}{s^2}$

Number of g’s is also sometimes called g-forces, however, I will not refer to number of g’s as g-Forces because number of g’s is not a force and calling it g-Forces will lead you to believe number of g’s is a force, but number of g’s is NOT a force.

Calculating the Number of g’s for an object at rest:



$$horizontal\ number\ of\ g's = \frac{a_x}{g_{Earth}} = \frac{0}{g_{Earth}} = 0$$

$$\sum F_y = F_N - F_g = ma_y = m(0) = 0 \Rightarrow F_N = F_g$$

$$vertical\ number\ of\ g's = \frac{F_N}{F_{g\ Earth}} = \frac{F_g}{F_{g\ Earth}} = 1$$

Calculating the Number of g’s for astronauts in the International Space Station:

$$horizontal\ number\ of\ g's = \frac{a_x}{g_{Earth}} = \frac{0}{g_{Earth}} = 0$$

$$vertical\ number\ of\ g's = \frac{F_N}{F_{g\ Earth}} = \frac{0}{F_{g\ Earth}} = 0 = apparent\ \ weightlessness$$

Apparent weightlessness is where the net Number of g’s acting on the object equals zero.

In the lecture “What is the Maximum Speed of a Car at the Top of a Hill?”, we determined the maximum linear velocity to drive a car over a hill and have the tires not leave the ground. When we did that problem, the critical point was where the force normal was equal to zero. In other words, the speed we determined is also the speed to drive the car such that the passengers would experience apparent weightlessness or where they would feel weightless while going over the hill.

Number of g’s is a ratio of the acceleration experienced by the object with respect to the acceleration we typically experience here on planet Earth. If you are experiencing 2 vertical g’s, you feel like you weigh twice what you normally do. If you are experiencing 1 and a half horizontal g’s, you feel like you have a horizontal weight which is 1 and a half times your normal vertical weight and realize you don’t normally experience weight horizontally. … Number of g’s gives us a way of comparing the acceleration an object is experiencing to what the object would normally experience here on planet Earth.

Other Number of g's examples:

2011 Toyota Prius: $v_i = 0,\ \Delta t = 5.43\,\text{sec},\ v_f = 56\dfrac{km}{hr}\times\dfrac{1mi}{1.609km} = 34.804 \approx 35\dfrac{mi}{hr},$

$$v_f = 56\frac{km}{hr}\times\frac{1hr}{3600\,\text{sec}}\times\frac{1000m}{1km} = 15.\overline{5}\frac{m}{s};\ a_x = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{\Delta t} = \frac{15.\overline{5} - 0}{5.43} = 2.86474\frac{m}{s^2}$$

$$\text{horizontal \# of g's} = \frac{a_x}{g_{Earth}} = \frac{2.86474}{9.81} = 0.29202 \approx 0.29$$

2020 Tesla Roadster:♥ $v_i = 0,\ \Delta t = 1.9\,\text{sec},\ v_f = 60\dfrac{mi}{hr}\times\dfrac{1.609km}{1mi} = 96.54 \approx 97\dfrac{km}{hr},$

$$v_f = 60\frac{mi}{hr}\times\frac{1hr}{3600\,\text{sec}}\times\frac{1609m}{1mi} = 26.81\overline{6}\frac{m}{s};\ a_x = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{\Delta t} = \frac{26.81\overline{6} - 0}{1.9} = 14.114\frac{m}{s^2}$$

$$\text{horizontal \# of g's} = \frac{a_x}{g_{Earth}} = \frac{14.144}{9.81} = 1.4387 \approx 1.4$$

NASA Space Shuttle:♣ "Astronauts normally experience a maximum g-force of around 3gs during a rocket launch."

Russian Soyuz:♠ "During a Soyuz launch, g-forces from 3.6 to 4.2g acceleration of gravity"

♥ https://blog.caranddriver.com/new-tesla-roadster-first-look-zero-to-60-in-1-9-seconds-250-mph-top-speed-620-mile-range/
♣ https://www.spaceanswers.com/space-exploration/what-g-force-do-astronauts-experience-during-a-rocket-launch/
♠ http://www.space-affairs.com/index.php?wohin=3rdfloor_p

Flipping Physics Lecture Notes:

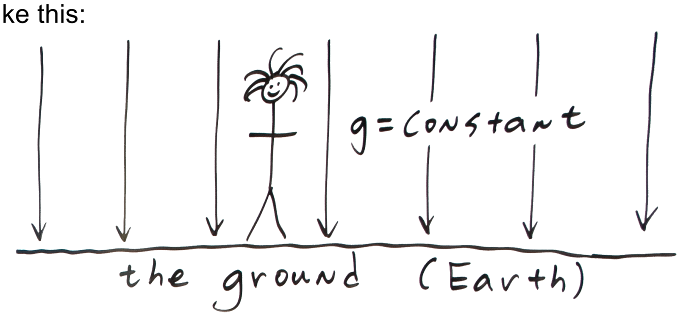
Gravitational Field Introduction

We have two equations for the force of gravity:

- $F_g = m_o g$ which is true when acceleration due to gravity is constant.
- Newton's Universal Law of Gravitation: $F_g = \frac{Gm_1 m_2}{r^2}$ which is always true.

The constant gravitational field equation is derived from the equation for the force of gravity when g is constant:

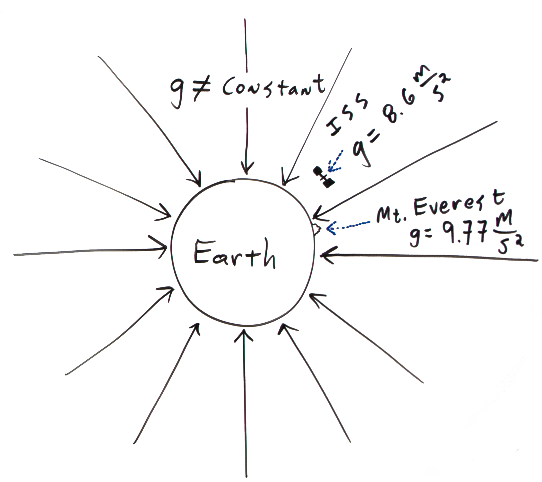
- $F_g = mg \Rightarrow g = \frac{F_g}{m}$ Constant gravitational field equation
- It has units of $\frac{N}{kg}$ which is the same as $\frac{m}{s^2}$. Because $g = \frac{F_g}{m} \Rightarrow \frac{N}{kg} = \frac{\frac{kg \cdot m}{s^2}}{kg} = \frac{m}{s^2}$
- We illustrate it like this:



- Lines are called Field Lines.
  - The strength of a field is illustrated by how close the field lines are to one another.
  - Closer field lines illustrates a stronger field.
  - The gravitational field lines in the above illustration are parallel to one another because the gravitational field is constant.

Gravitational field around any object is derived from Newton's Universal Law of Gravitation:

- $F_g = m_o g = \frac{Gm_o m_E}{r^2} \Rightarrow g = \frac{Gm_E}{r^2}$ This is around the Earth.
- $g = \frac{Gm}{r^2}$ But all objects with mass are surrounded by a gravitational field.
- Notice in the illustration the field lines are farther apart the farther from the object. This is because the strength of the gravitational field decreases as we move farther from the planet.



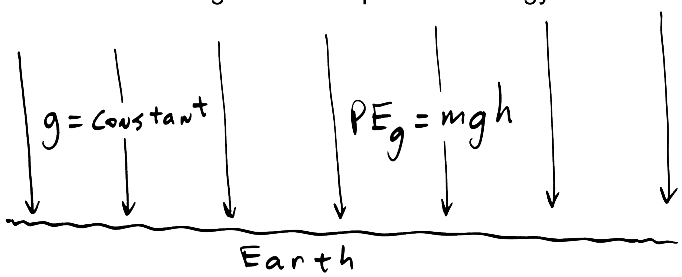
Gravitational fields caused by single objects are always directed towards the center of mass of the object. On the surface of the Earth, that means *down*.

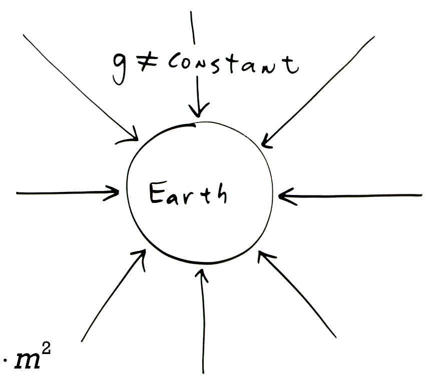
Flipping Physics Lecture Notes:

Universal Gravitation Potential Energy Introduction

Previous to today our only equation for gravitational potential energy was $PE_g = mgh$.

- h is the vertical height above the horizontal zero line.
- Remember to set your horizontal zero line
- This equation is true when the acceleration due to gravity is constant. Like on the surface of planet Earth, where we are for the majority of our lives.
- This is the gravitational potential energy which exists between an object and the planet.





Universal Gravitational Potential Energy is $U_g = -\frac{Gm_1m_2}{r}$.

- G is the Universal Gravitational Constant: $G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$
- m_1 and m_2 are the two masses which have the gravitational potential energy.
- r is the distance between the centers of mass of the two objects.
- This equation is always true for the gravitational potential energy which exists between two objects.
- This has a predetermined zero line where the two objects are infinitely far apart, r = ∞. A reason the zero line is located infinitely far away is to make Newton's Universal Law of Gravitation and universal gravitational potential energy both be zero when the objects are at the same locations.

It is helpful to compare these two equations to the two we have for the force of gravity:

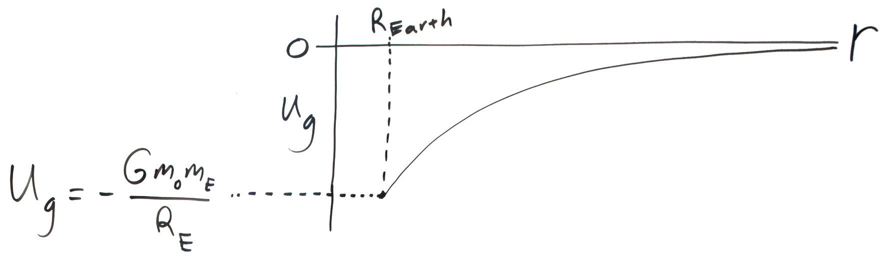
- $F_g = m_o g$ which is true on the surface of a planet. (Comparable to when we use $PE_g = mgh$)
- $F_g = \frac{Gm_1m_2}{r^2}$ which is always true. (Comparable to when we use $U_g = -\frac{Gm_1m_2}{r}$)

The gravitational potential energy which exists between an object and the Earth:

- When the object is on the surface of the planet: $U_g = -\frac{Gm_o m_E}{R_E}$
  - Also because $g_{Earth} = \frac{Gm_E}{R_E^{\;2}}$ ♥ at r = R_E,

    $$U_g = -\frac{Gm_o m_E}{R_E} = \left(-\frac{Gm_o m_E}{R_E}\right)\left(\frac{R_E}{R_E}\right) = -\left(\frac{Gm_E}{R_E^{\;2}}\right) m_o R_E = -m_o g R_E$$

    which is remarkably like $PE_g = mgh$.
- When the object is infinitely far from the planet: $U_g = -\frac{Gm_o m_E}{\infty} = 0$
- Between r = R_E and r = ∞ the shape of the graph is concave down as shown below.



Three things students need to be careful of:

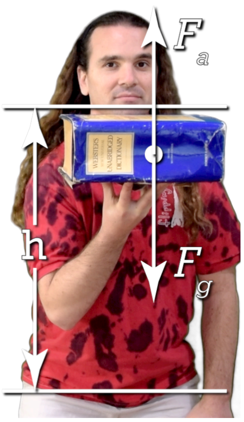
1) Please do not forget the negative in $U_g = -\frac{Gm_1 m_2}{r}$.
2) Gravitational potential energy requires two objects. In other words, one object cannot have gravitational potential energy by itself.
3) The variable "r" is ***not*** squared in universal gravitational potential energy. It is in Newton's Universal Law of Gravitation, which means students have a tendency to add a square, it's not there, beware.

♥ Previously derived in "Deriving the Acceleration due to Gravity on any Planet and specifically Mt. Everest". http://www.flippingphysics.com/mount-everest-gravity.html

Flipping Physics Lecture Notes:

Deriving the Binding Energy of a Planet

All objects on planet Earth are gravitationally bound to it. Jump up and you will discover it is very difficult to leave the planet. However, if you are given enough mechanical energy, you will be able to leave the planet. The minimum amount of energy or work necessary to completely remove an object from another object is called the “binding energy.” One can also determine the “binding energy” of atoms and atomic particles. In this lesson we are determining the binding energy of a planet.



In order to determine an object’s binding energy, let’s start by analyzing lifting a book at a constant velocity a short, vertical distance, h:

$$\sum F_y = F_a - F_g = ma_y = m(0) = 0 \Rightarrow F_a = F_g = mg$$

Because the book is moving at a constant velocity, the force applied to the book and the force of gravity acting on the book are equal in magnitude.

$$W = Fd\cos\theta \Rightarrow W_{F_a} = F_a d\cos\theta = (mg)(h)\cos(0) = mgh$$

The work done by the force applied equals mass times acceleration due to gravity times h which is, in this case, the change in height of the book. Notice mgh is the value for the gravitational potential energy in a constant gravitational field. Also, θ is the angle between the direction of the force applied and the displacement of the book, and is therefore zero degrees.

If we set the horizontal zero line at the initial height of the book, then $h_i = 0$ and $h_f = h$ then:

$$\Delta PE_g = PE_{gf} - PE_{gi} = mgh_f - mgh_i = mgh - mg(0) = mgh$$

Therefore the work done by the force applied on the book equals the change in gravitational potential energy of the book: $W_{F_a} = \Delta PE_g$

Remember the binding energy of a planet is the minimum amount of energy or work necessary to completely remove an object from the planet. In this case the gravitational potential energy we need to use is $U_g = -\frac{Gm_1m_2}{r}$ because the gravitational field is not constant.

$$W_{F_a} = \Delta U_g = U_{gf} - U_{gi} = -\frac{Gm_om_E}{r_f} - \left(-\frac{Gm_om_E}{r_i}\right) = -\frac{Gm_om_E}{\infty} + \frac{Gm_om_E}{R_E} = \boxed{\frac{Gm_om_E}{R_E}}$$

Because the object is being completely removed from the planet, the final position is infinitely far away from the planet, $r_f = \infty$, and anything divided by infinity equals zero.

Flipping Physics Lecture Notes:

Deriving Escape Velocity

Example: Determine the escape velocity of planet Earth. Assume no air resistance and no planet rotation.
Knowns: $m_{Earth} = 5.97 \times 10^{24}\,kg;\ R_{Earth\ average} = 6.37 \times 10^{6}\,m$ [♥]

The escape velocity of a planet is the minimum speed at which an object can be launched such that it would take an infinite amount of time to slow the object to a stop.

No force applied and no air resistance means we can use conservation of mechanical energy: $ME_i = ME_f$

Set the initial point at the surface of the planet and the final point infinitely far away from the planet.

The object starts with gravitational potential energy and kinetic energy. It ends with zero gravitational potential energy, because it is infinitely far away from the planet. It also ends with zero kinetic energy because this is the *minimum speed*. If the speed were anything greater, the object would end with some kinetic energy.

$$\Rightarrow U_{gi} + KE_i = 0$$

Now we can substitute in equations.

$$\Rightarrow -\frac{Gm_1m_2}{r} + \frac{1}{2}mv_i^{\,2} = 0$$

Add gravitational potential energy to both sides and substitute in variables for mass, radius, and velocity.

$$\Rightarrow \frac{1}{2}m_o v_{oi}^{\,2} = \frac{Gm_o m_E}{R_E}$$

Everyone brought the mass of the object to the party and solve for the velocity of the object initial, which is the escape velocity.

$$\Rightarrow \frac{1}{2}v_{oi}^{\,2} = \frac{Gm_E}{R_E} \Rightarrow v_{oi} = v_{escape} = \sqrt{\frac{2Gm_E}{R_E}}$$

Please notice the mass of the object is irrelevant.

Substitute in numbers.

$$\Rightarrow v_{escape} = \sqrt{\frac{2\left(6.67\times10^{-11}\right)\left(5.96\times10^{24}\right)}{6.37\times10^{6}}} = 11{,}181.38\frac{m}{s}\times\frac{1km}{1000m} \approx \boxed{11.2\frac{km}{s}}$$

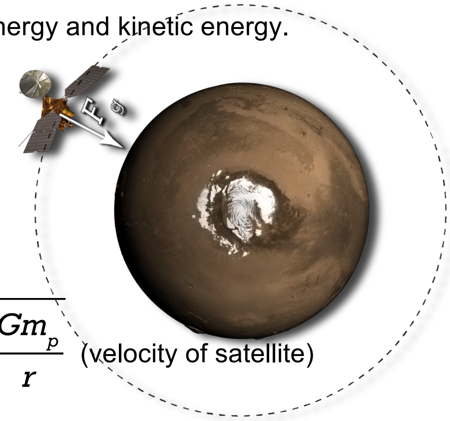
$$\Rightarrow v_{escape} = 11{,}181.38\frac{m}{s}\times\frac{3600s}{1hr}\times\frac{1mi}{1609m} = 25{,}017.38 \approx 25{,}000\frac{mi}{hr} \approx \boxed{2.50\times10^{4}\frac{mi}{hr}}$$

♥ https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html

Flipping Physics Lecture Notes:

Mechanical Energy of a Satellite

A satellite is an object in orbit which has both gravitational potential energy and kinetic energy.



$$ME_{total} = U_g + KE = -\frac{Gm_s m_p}{r} + \frac{1}{2} m_s v_s^{\ 2}$$

In order to simplify this expression we need to identify that the only force acting on the satellite is the force of gravity which is directed toward the center of mass of the planet. Therefore we can sum the forces in the in-direction on the satellite.

$$\sum F_{in} = F_g = ma_c \Rightarrow \frac{Gm_s m_p}{r^2} = m_s \frac{v_s^{\ 2}}{r} \Rightarrow \frac{Gm_p}{r} = v_s^{\ 2} \Rightarrow v_s = \sqrt{\frac{Gm_p}{r}}$$ (velocity of satellite)

(everybody brought the mass of the satellite divided by the radius to the party)

Substitute $v_s^{\ 2} = \frac{Gm_p}{r}$ into the ME_{total} equation:

$$ME_{total} = -\frac{Gm_s m_p}{r} + \frac{1}{2} m_s \frac{Gm_p}{r} = \frac{Gm_s m_p}{r}\left(-1 + \frac{1}{2}\right) = \boxed{\left(-\frac{1}{2}\right)\frac{Gm_s m_p}{r}}$$

That's right, the total mechanical energy of a satellite equals half the universal gravitational potential energy between the satellite and the planet: $ME_{total} = \left(-\frac{1}{2}\right)\frac{Gm_s m_p}{r} = \frac{1}{2}\left(-\frac{Gm_s m_p}{r}\right) = \frac{1}{2}U_g$

Realize, because Universal Gravitational Mechanical energy is always negative, the Total Mechanical Energy is still negative.

I can't help but point out this out about the escape velocity we determined in a previous lesson:

$$v_{escape} = \sqrt{\frac{2Gm_p}{r}} = \sqrt{2}\sqrt{\frac{Gm_p}{r}} = \left(\sqrt{2}\right)v_{satellite}$$

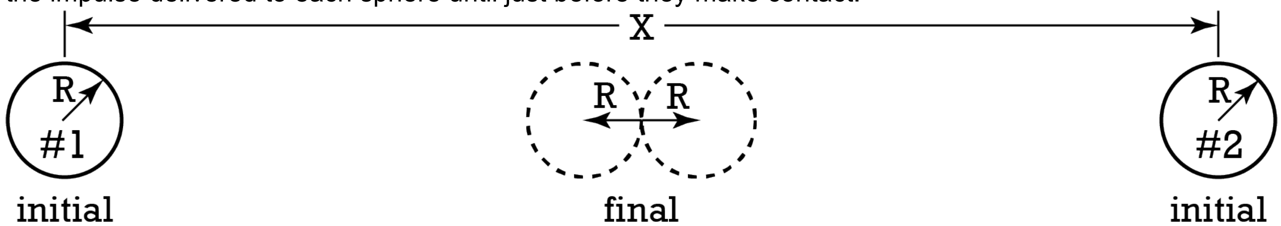
That is correct, the escape velocity equals $\sqrt{2}$ times the satellite velocity.
I don't know why that is, or why it is interesting, however, it is interesting.

FYI: The free body diagram picture shows NASA's Mars Reconnaissance Orbiter in orbit around Mars.

Flipping Physics Lecture Notes:

Impulse for Two Objects being Attracted to One Another

Example: In a universe devoid of anything else, two identical spheres of mass, m, and radius, R, are released from rest when they have a distance between their centers of mass of X. Find the magnitude of the impulse delivered to each sphere until just before they make contact.



Knowns: $v_{1i} = v_{2i} = 0;\ m_1 = m_2 = m;\ r_1 = r_2 = R;\ \text{initial center of mass distance} = X;\ J = ?$

The equation for Impulse is: $\vec{J} = \Delta\vec{p} = \text{Area under Force vs. Time curve}$

Area under Force vs. Time curve is unhelpful in this situation. (There is no Force vs. Time curve.)

$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i = m\vec{v}_f - m\vec{v}_i = m\vec{v}_f - m(0) = m\vec{v}_f$$

All we need is the final velocity of one sphere. In fact, we only need the magnitude of the final velocity.

No work done by a Force Applied and no energy converted to heat or sound via friction → Conservation of Mechanical Energy: $ME_i = ME_f$; The initial and final points are already identified in the drawing. The only type of initial mechanical energy is the universal gravitational potential energy. Finally both objects have kinetic energy and universal gravitational potential energy.

$$\Rightarrow U_{gi} = KE_{1f} + KE_{2f} + U_{gf}$$

Substitute in equations and realize one expression for the universal gravitational potential energy includes the universal gravitational potential energy for both spheres.

$$\Rightarrow -\frac{Gm_1m_2}{r_i} = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2 - \frac{Gm_1m_2}{r_f}$$

Substitute in values: $r_i = X;\ m_1 = m_2 = m;\ v_{1f} = v_{2f} = v_f;\ r_f = 2R$ (because the final distance between the centers of mass includes two radii.) Everybody brought mass to the party!

$$\Rightarrow -\frac{Gmm}{X} = \frac{1}{2}mv_f^2 + \frac{1}{2}mv_f^2 - \frac{Gmm}{2R} \Rightarrow -\frac{Gm}{X} = \frac{1}{2}v_f^2 + \frac{1}{2}v_f^2 - \frac{Gm}{2R}$$

And now some algebra. $\Rightarrow \frac{Gm}{2R} - \frac{Gm}{X} = v_f^2 = Gm\left(\frac{1}{2R} - \frac{1}{X}\right) \Rightarrow v_f = \sqrt{Gm\left(\frac{1}{2R} - \frac{1}{X}\right)}$

And back to the equation for the magnitude of the impulse.

$$J = mv_f = m\sqrt{Gm\left(\frac{1}{2R} - \frac{1}{X}\right)} = \boxed{\sqrt{Gm^3\left(\frac{1}{2R} - \frac{1}{X}\right)}}$$

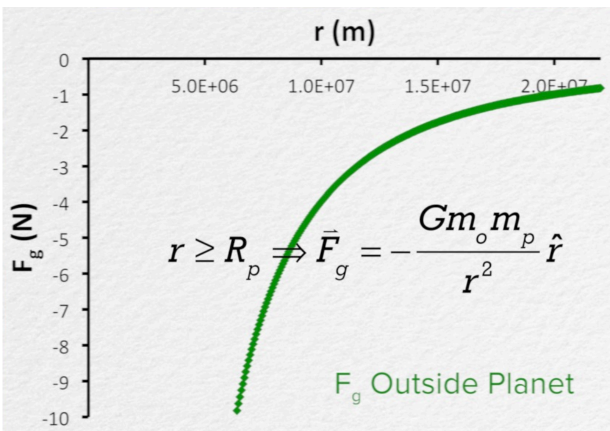
Flipping Physics Lecture Notes:

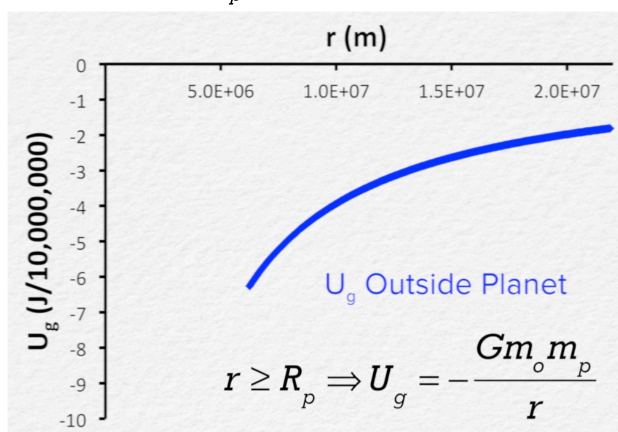
Force of Gravity and Gravitational Potential Energy Functions from Zero to Infinity (but not beyond)

We already used Newton's Universal Law of Gravitation to determine the equation for Universal Gravitational Potential Energy. We did that by using the equation that relates a conservative force to the potential energy associated with that conservative force: $F_x = -\frac{dU}{dx}$. Remember, this equation is ***not*** on the AP Physics C: Mechanics equation sheet:

$$\vec{F}_g = -\frac{Gm_o m_p}{r^2}\hat{r} \;\&\; F_x = -\frac{dU}{dx} \Rightarrow U_g = -\frac{Gm_1 m_2}{r}$$

If we specify the two masses to be a planet and an object, we can draw the graphs of the force of gravity and gravitational potential energy as a function of position, r, where $r \geq R_p$





Notice the force of gravity is proportional to $\frac{1}{r^2}$, whereas the universal gravitational potential energy is proportional to $\frac{1}{r}$.

We also want to determine the universal gravitational potential energy *inside* the planet, where $r \leq R_p$. In order to determine this we first need the force of gravity *inside* the planet. In order to do that we need to make some assumptions:

1) The planet has a constant density, ρ.
    a. Yes, this is not actually true, but it is a good thought experiment.
2) We need to determine the force of gravity acting on an object that can move, without friction, through a tunnel we have drilled all the way through the center of the planet.
    a. And you thought assuming the planet had a constant density was a stretch…
3) In order to do this, we need to assume the planet is not rotating.
4) We need to assume the only mass of the planet that exerts a force of gravity on our object as $r < R_p$ is the mass of the planet that is inside a hypothetical sphere created by our variable r, the radius of the current location of the object. In other words, while all the mass of outside of r still causes a force of gravity on the object, due to symmetry and the fact that the force of gravity is proportional to the inverse square of the distance, all of the forces of gravitational attraction caused by the mass of the planet with $r \geq R_p$ cancel out. This is called Newton's Shell Theorem♥, in case you were curious.

♥ https://www.math.ksu.edu/~dbski/writings/shell.pdf

Here is the derivation of the force of gravitational attraction between an object and a planet as the object moves below the surface of the planet:

$$\rho = \frac{m_{total}}{\forall_{total}} = \frac{m_{in}}{\forall_{in}} \Rightarrow m_{in} = \rho \forall_{in} = \rho\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi\rho r^3$$

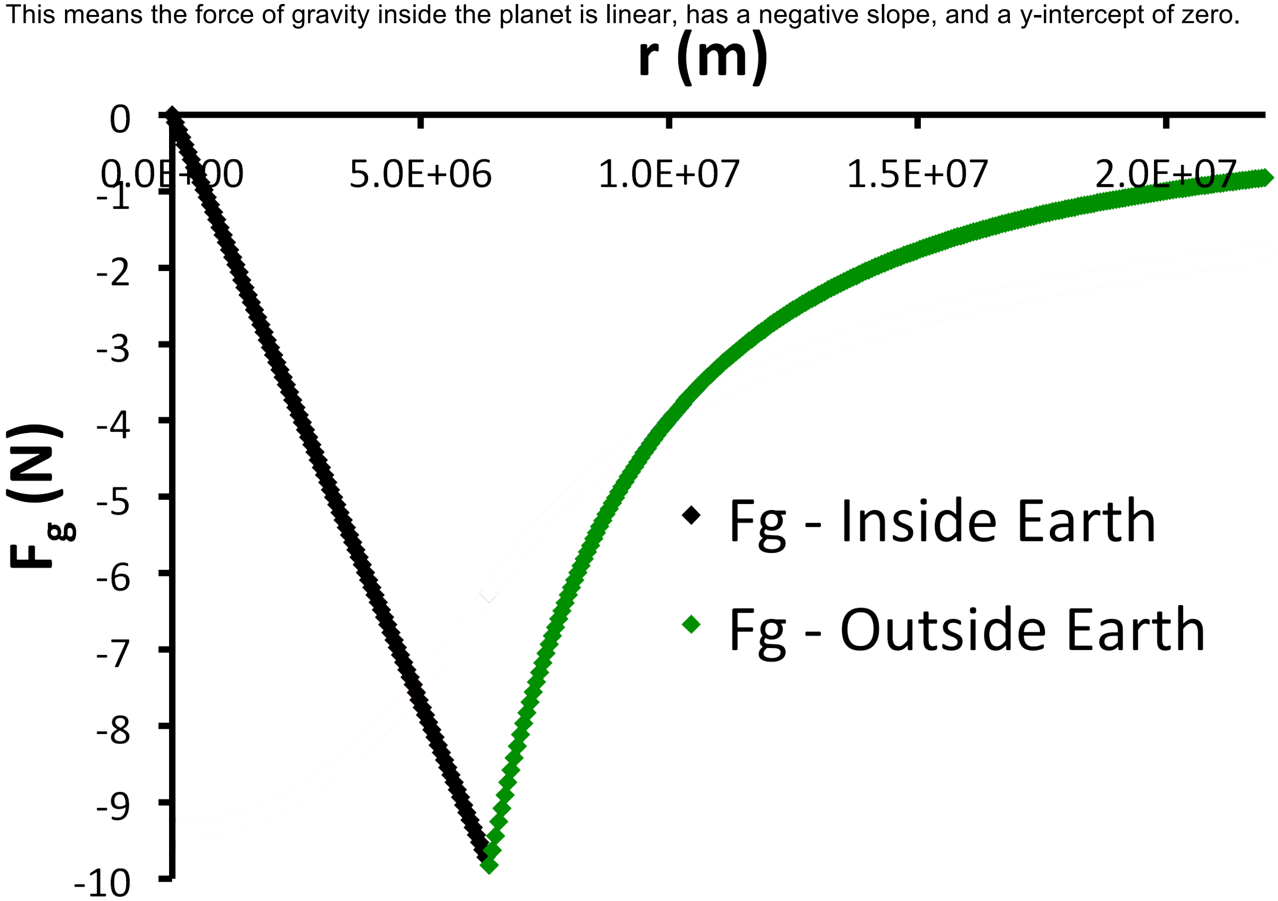
Note: m_{in} and $\forall_{in}$ refer to the mass and volume that are inside the sphere created by the variable r.

$$F_{g_{in}} = \frac{G m_o m_{in}}{r^2} = \frac{G m_o}{r^2}\left(\frac{4}{3}\pi\rho r^3\right) = \left(\frac{4\pi G m_o \rho}{3}\right) r$$

Notice $\frac{4\pi G m_o \rho}{3}$ are all constants.

In terms of vectors, the force of gravity is directed towards the center of the planet: $\vec{F}_{g_{in}} = -\left(\frac{4\pi G m_o \rho}{3}\right) r\ \hat{r}$

This means the force of gravity inside the planet is linear, has a negative slope, and a y-intercept of zero.



Now we can start working with energy and the conservative force equation:

$$F_x = -\frac{dU}{dx} \Rightarrow F_{g_{in}} = -\frac{dU_{g_{in}}}{dr} \Rightarrow dU_{g_{in}} = -F_{g_{in}}\,dr \Rightarrow \int dU_{g_{in}} = -\int F_{g_{in}}\,dr \Rightarrow U_{g_{in}} = -\int -\left(\frac{4\pi G m_o \rho}{3}\right) r\,dr$$

$$\Rightarrow U_{g_{in}} = \left(\frac{4\pi G m_o \rho}{3}\right)\int r\,dr = \left(\frac{4\pi G m_o \rho}{3}\right)\frac{r^2}{2} + C = \left(\frac{2\pi G m_o \rho}{3}\right) r^2 + C$$

To solve for C, we can use the fact that U_g has to have the same value for both expressions where $r = R_p$:

$$U_g\left(@\, r = R_p\right) = \left(\frac{2\pi G m_o \rho}{3}\right) R_p^{\ 2} + C = -\frac{G m_o m_p}{R_p} \Rightarrow C = -\frac{G m_o m_p}{R_p} - \left(\frac{2\pi G m_o \rho}{3}\right) R_p^{\ 2}$$

$$\Rightarrow U_{g_{in}} = \left(\frac{2\pi G m_o \rho}{3}\right) r^2 - \left(\frac{G m_o m_p}{R_p} + \left(\frac{2\pi G m_o \rho}{3}\right) R_p^{\ 2}\right)$$

I know this looks confusing, however, the only variable in this whole equation is r, everything else is a constant. This equation is essentially $U_{g_{in}} = \# r^2 - \#$. It is a parabolic function shifted down on the y-axis.

The graph starts at a large negative number, is proportional to r^2 until $r = R_{Earth}$ and then is proportional to 1/r:

For $r \le R_p$, $\vec{F}_{g_{in}} = -\left(\frac{4\pi G m_o \rho}{3}\right) r\,\hat{r}$ and for $r \ge R_p$, $\vec{F}_g = -\frac{G m_o m_p}{r^2}\hat{r}$

For $r \le R_p$, $U_{g_{in}} = \left(\frac{2\pi G m_o \rho}{3}\right) r^2 - \left(\frac{G m_o m_p}{R_p} + \left(\frac{2\pi G m_o \rho}{3}\right) R_p^{\ 2}\right)$ and for $r \ge R_p$, $U_g = -\frac{G m_o m_p}{r}$

A 1-kilogram mass with respect to the Earth has graphs which looks like this: