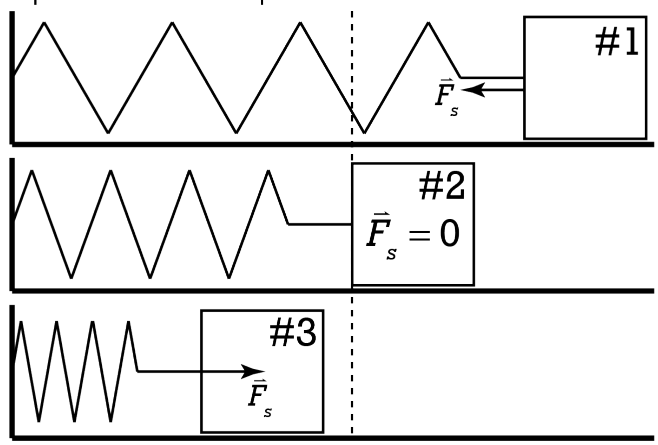
Flipping Physics Lecture Notes:

Simple Harmonic Motion Introduction via a Horizontal Mass-Spring System

A Horizontal Mass-Spring System is where a mass is attached to a spring, oriented horizontally, and then placed on a frictionless surface. When the mass is at rest, the spring will be at the equilibrium or rest position. This is the vertical line shown in the diagram below. We can then pull the mass to the right and hold it there. Let's call this position #1. Notice at position #1 the force of the spring, $\vec{F}_{s1}$, is to the left because the displacement of the spring from equilibrium position, $\vec{x}_1$, is to the right. We know this because of Hooke's Law, $\vec{F}_s = -k\vec{x}$.

When we let go of the mass the spring force will accelerate the mass to the left, the mass will pass through the equilibrium position, which we can call position #2. After passing through rest position, the mass will pause to the left of the equilibrium position. Let's call this position #3. Notice position #3 is the same distance from position #2 as position #1.



At position #2, the displacement from equilibrium position, $\vec{x}_2$, is zero. Therefore, according to Hooke's Law, the force of the spring, $\vec{F}_{s2}$, is also equal to zero.

At position #3, the displacement from rest position, $\vec{x}_3$, is to the left. Therefore, according to Hooke's Law, the force of the spring, $\vec{F}_{s3}$, is to the right.

In the absence of friction, the spring will continue to move back and forth through these positions like this: 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, 2, 1 … and it will never stop.

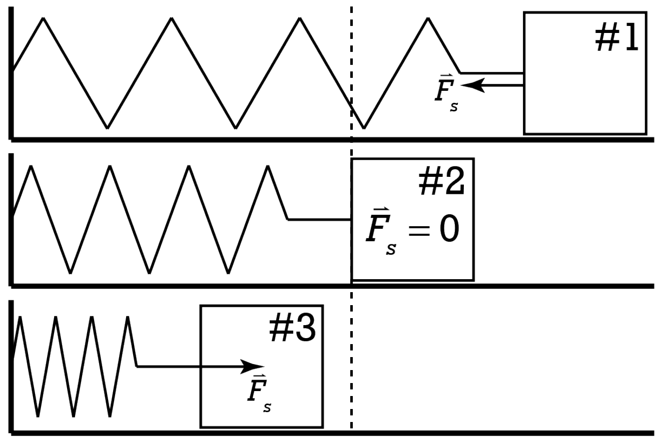
This is called Simple Harmonic Motion. There are two requirements for the force that causes simple harmonic motion:

1) It must be a Restoring Force: A force that is always towards equilibrium.
    a. The spring force is a restoring force because it is always directed toward rest position and therefore will always accelerate the mass toward equilibrium position.
2) The force must be proportional to displacement from equilibrium position.
    a. According to Hooke's Law, $\vec{F}_s = -k\vec{x}$, the spring force is proportional to displacement from equilibrium position. In other words, the larger the displacement from equilibrium position, the larger the spring force.

Flipping Physics Lecture Notes:

Simple Harmonic Motion - Force, Acceleration, and Velocity at 3 Positions

We previously defined three locations for an object in simple harmonic motion. Positions 1 and 3 are at the maximum displacement from and on either side of equilibrium position. Position 2 is when the mass is at rest position. Now let's determine some basics about the *magnitudes* of the velocities and accelerations at those positions.



Notice that at positions 1 and 3, the velocity of the mass changes directions. This means the velocities at 1 and 3 are zero. This is just like the velocity at the top is zero for an object in free fall. This means the magnitude of the velocity halfway in between those two positions, in other words at position 2, will have a maximum value.

At positions 1 and 3, displacement from equilibrium position, x, will have a maximum magnitude. That means, according to Hooke's Law, $\vec{F}_s = -k\vec{x}$, the spring force will also have it's maximum magnitude at positions 1 and 3. If we sum the forces in the x direction, $\sum F_x = F_s = ma_x$, we can see the acceleration will also have its maximum magnitude at positions 1 and 3.

Please realize that the spring force changes as a function of position, therefore, the net force in the x-direction changes as a function of position, therefore the acceleration of the mass changes as a function of position, therefore simple harmonic motion is not uniformly accelerated motion. In simple harmonic motion the **acceleration is not constant**, therefore, you **cannot** use the uniformly accelerated motion equations.

$$\sum F_x = F_s = ma_x \Rightarrow kx = ma_x \Rightarrow x \neq \text{constant} \text{ therefore } a \neq \text{constant}.^{\heartsuit}$$

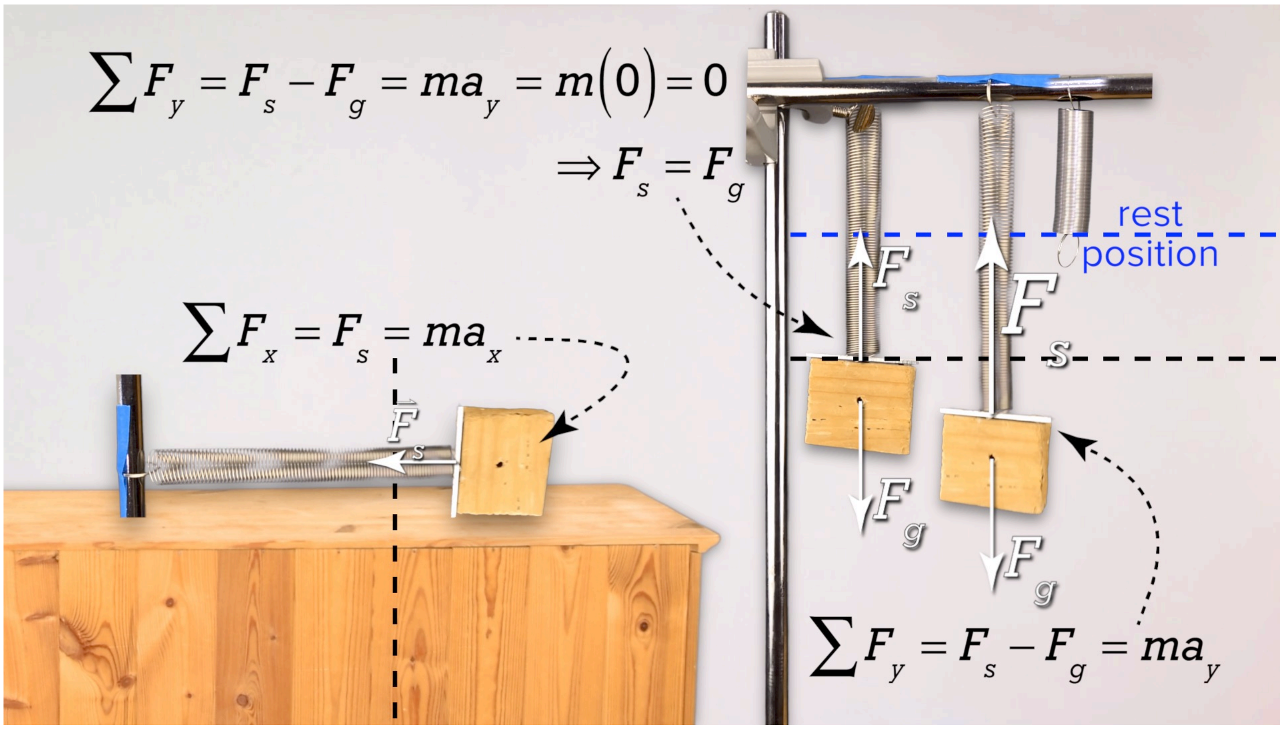
♥ Yes, I am ignoring whether the spring force is to the left or right in this equation. It does not matter. I am simply showing that simple harmonic motion is **not** uniformly accelerated motion.

Flipping Physics Lecture Notes:

Horizontal vs. Vertical Mass-Spring System

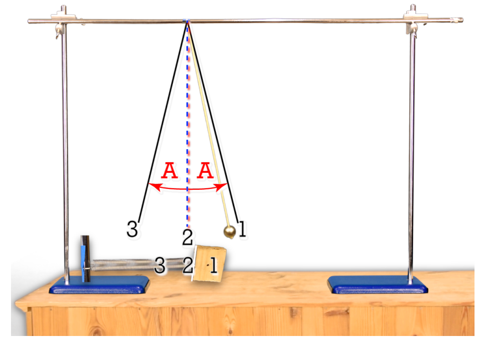
Horizontal and vertical mass-spring systems are both in simple harmonic motion.

- A vertical mass spring system oscillates around the point where the downward force of gravity and the upward spring force cancel one another out.
- The restoring force for a horizontal mass-spring system is just the spring force, because that is the net force in the x-direction.
- The restoring force for a vertical mass-spring system is the net force in the y-direction which equals the spring force minus the force of gravity.



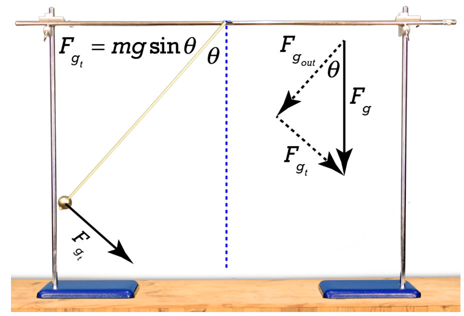
Flipping Physics Lecture Notes:

When is a Pendulum in Simple Harmonic Motion?



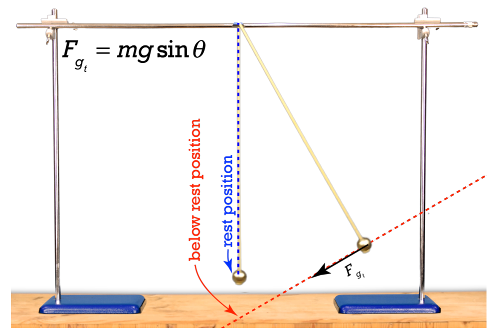
Mass-spring systems and pendulums are both in simple harmonic motion. Both oscillate around an equilibrium position and have a restoring force pointed towards the equilibrium position that increases proportionally with displacement from the equilibrium or rest position.

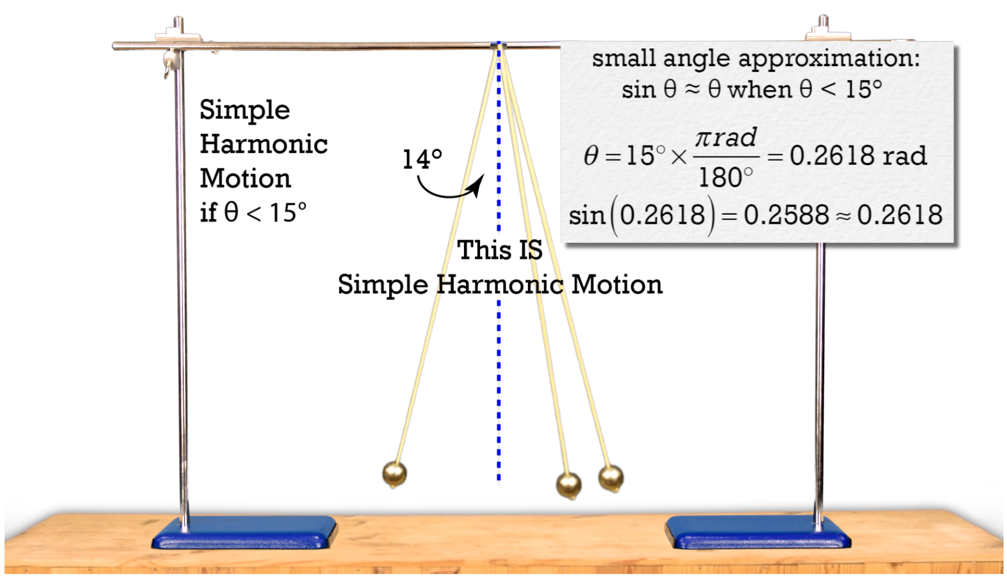
- The displacement from equilibrium position for a pendulum is an *angular* displacement.
    - Units are in degrees or radians.
    - Symbol is theta, θ.
    - Maximum displacement from equilibrium position is still Amplitude, A.



- The restoring force for a pendulum is the force of gravity tangential to the path of the pendulum. This force is:
    - Proportional to displacement from equilibrium position and
    - Directed toward equilibrium position.

Actually, the force of gravity tangential is only considered to be directed toward equilibrium or rest position for "small angles". Typically I consider this to be less than 15°, however, some sources require the angle to be less than 10°. It depends on how much error you are willing to allow. The larger the angle, the larger the error. This is because of the *small angle approximation.*





Flipping Physics Lecture Notes:

Demonstrating What Changes the Period of Simple Harmonic Motion

The period of simple harmonic motion is the time it takes to complete one full cycle.

- The units for period are typically seconds or seconds per cycle, however, they could also be in minutes, hours, days, fortnights, decades, millenniums, etc.
- The symbol for period is T.

Using our previously defined postions of 1 and 3 where the object is at its maximum displacement from equilibirum position and position 2 is at the equilibrium position, recall that the simple harmonic motion pattern is 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, 2, 1, 2, 3, …

One full cylce in terms of position could be:

- 1, 2, 3, 2, 1 or 3, 2, 1, 2, 3 or 2, 1, 2, 3, 2 or 2, 3, 2, 1, 2
- or starting and ending somewhere between one of the positions as long as:
    1. the object starts and ends at the same location
    2. the object is moving in the same direction at the end as at the start

The equations for the period of simple harmonic motion are:

- For a mass-spring system: $T = 2\pi\sqrt{\frac{m}{k}}$
    - m is the mass in the mass-spring system.
    - k is the spring constant of the spring.
- For a pendulum: $T = 2\pi\sqrt{\frac{L}{g}}$
    - L is the “pendulum length” which is the distance from the center of suspension to the center of mass of the pendulum bob.
        - Center of suspension is the top, fixed end of the pendulum.
        - Pendulum bob is the mass at the bottom of the pendulum.
    - g is the acceleration due to gravity in which the pendulum is located.
        - On Earth that would be 9.81 m/s^2.
    - This is called a *simple pendulum*. Meaning the rod/string is of negligible mass therefore the center of mass of a simple pendulum is the center of mass of the pendulum bob.

What affects the period of a pendulum and a mass-spring system?

- Amplitude is ***not*** in either period equation.
    - Amplitude does ***not*** affect the period of a pendulum or the period of a mass-spring system.
- Acceleration due to gravity is ***not*** in the period equation for a mass-spring system.
    - g does ***not*** affect the period of a mass-spring system.
- Mass is ***not*** in the period equation for a pendulum.
    - The mass of the pendulum bob does ***not*** affect the period of a pendulum.
- Increasing the mass in a mass-spring system increases its period.
    - $m\uparrow \Rightarrow T\uparrow$ for a mass-spring system.
- Increasing the spring constant in a mass-spring system decreases its period.
    - $k\uparrow \Rightarrow T\downarrow$ for a mass-spring system.
- Increasing the “pendulum length” increases its period.
    - $L\uparrow \Rightarrow T\uparrow$ for a pendulum.
- Increasing the acceleration due to gravity decreases the period of a pendulum.
    - $g\uparrow \Rightarrow T\downarrow$ for a pendulum.

Flipping Physics Lecture Notes:

Triple the Mass in a Mass-Spring System. How does Period Change?

Example: If the mass in a mass-spring system is tripled, how does the period change?

Knowns: $m_2 = 3m_1$ and $T_2 = ?T_1$

We know the equation for the period of a mass-spring system: $T = 2\pi\sqrt{\frac{m}{k}}$

So the period of the original mass-spring system is: $T_1 = 2\pi\sqrt{\frac{m_1}{k}}$

And the period of the new mass-spring system with three times the mass is:

$$T_2 = 2\pi\sqrt{\frac{m_2}{k}} = 2\pi\sqrt{\frac{3m_1}{k}} = 2\pi\sqrt{\frac{m_1}{k}}\sqrt{3} = T_1\sqrt{3} \Rightarrow T_2 = T_1\sqrt{3}$$

So tripling the mass, increases the period by the square root of 3: $\boxed{T_2 = T_1\sqrt{3}}$

Demonstration: $T_1 = 1.67\text{s}$ & $T_2 = T_1\sqrt{3} = (1.67)\sqrt{3} = 2.8925 \approx 2.89\text{s}$

However, the observed value for the period with three times the mass is 2.83 seconds.

$$E_r = \frac{O-A}{A}\times 100 = \frac{2.83-2.8925}{2.8925}\times 100 = -2.1608 \approx -2.16\%$$

I think we can confidently say, “The physics works!!”

Flipping Physics Lecture Notes:

Frequency vs. Period in Simple Harmonic Motion

We have already defined the period, T, of simple harmonic motion as the time it takes for one full cycle or oscillation. Frequency, f, is defined as the number of cycles or oscillations per second. Hopefully you recognize then that frequency and period are inverses of one another.

$$\boxed{T = \frac{1}{f}}$$

The units for frequency are $\frac{cycles}{second}$ which we call hertz (Hz) after the 19th century German physicist Heinrich Hertz[♥] (1857-1894) who was the first to give conclusive proof of the existence of electromagnetic waves which were theorized by James Clerk Maxwell's electromagnetic theory of light which we will learn about later.

For example, if we have a vertical mass-spring system with a period of 0.77 seconds, the frequency of that mass-spring system is:

$$f = \frac{1}{T} = \frac{1}{0.77} = 1.2987 \approx 1.3 \frac{cycles}{second} \; or \, 1.3 \; Hz$$

Which means the mass-spring system should go through 1.3 oscillations every second.

Another example, if we have a pendulum which goes through 15 cycles in 11 seconds, then the frequency of that pendulum is:

$$f = \frac{15 \; cycles}{11 \; seconds} = 1.\overline{36} \approx 1.4 \text{ Hz}$$

Which we can compare to the period of the pendulum:

$$T = \frac{1}{f} = \frac{1}{1.36363} = 0.733333 \approx 0.73 \sec$$

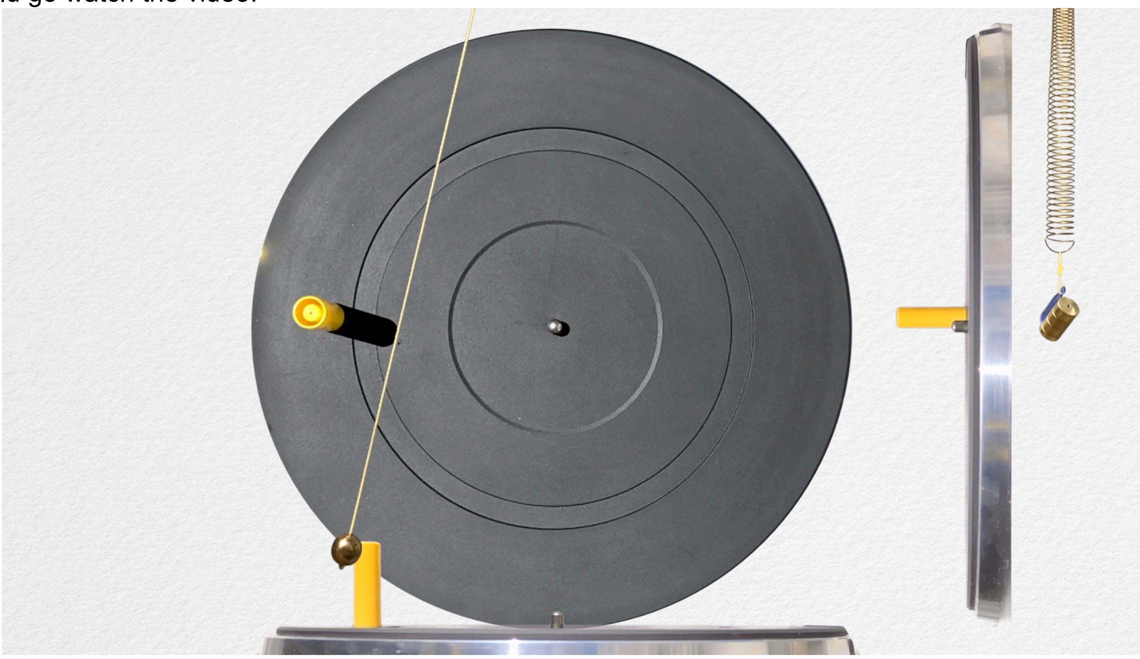
♥ https://commons.wikimedia.org/wiki/File:Heinrich_Rudolf_Hertz.jpg

Flipping Physics Lecture Notes:

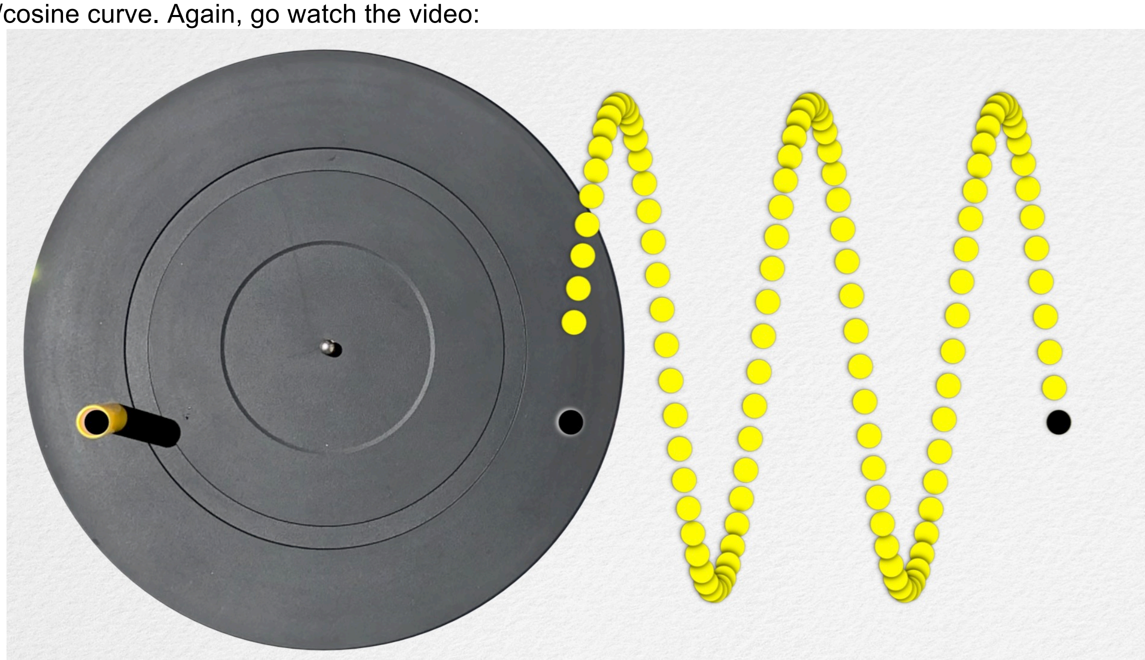
Comparing Simple Harmonic Motion to Circular Motion

https://www.flippingphysics.com/shm-vs-cm.html

Circular motion, when viewed from the side, is simple harmonic motion. It is difficult to see on paper, which is why I make the videos. If you just look at the x or y direction motion of the yellow marker cap on top of the rotating turntable, the cap is moving in simple harmonic motion. Here is a picture, however, really, you should go watch the video:



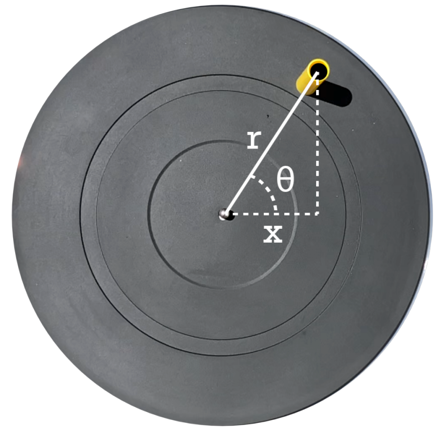
If you look at the location of the cap in one dimension as a function of time, then you end up with a sine/cosine curve. Again, go watch the video:



Flipping Physics Lecture Notes:

Simple Harmonic Motion – Position Equation Derivation

Circular motion, when viewed from the side, is simple harmonic motion. We can use this fact to derive an equation for the position of an object in simple harmonic motion.



- r is radius of the circular motion.
- x is the position of the cap in the x-direction, assuming the center of the turntable is the center of our coordinate system.
- θ is the angular displacement of the cap from an initial position where the cap was at its extreme position to the right.

$$\cos\theta = \frac{A}{H} = \frac{x}{r} \Rightarrow x = r\cos\theta \;\; \&$$

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\theta - 0}{t - 0} = \frac{\theta}{t} \Rightarrow \theta = \omega t$$

- Assuming we let $\theta_i = 0; t_i = 0; \theta_f = \theta; t_f = t$

$$x = r\cos\theta = r\cos(\omega t) \;\; \& \;\; \omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} = 2\pi f \qquad \text{remember: } f = \frac{1}{T}$$

Therefore $\theta = \omega t = 2\pi ft$ and $x = r\cos\theta = r\cos(\omega t) = r\cos(2\pi ft)$

Identifying that the maximum displacement from equilibrium position is the amplitude, A, is also r in the position equation. Therefore: $x(t) = A\cos(2\pi ft)$

Notice this is an equation which can be used to describe an object oscillating in simple harmonic motion. The equation could also be: $x(t) = A\sin(2\pi ft)$ or even $x(t) = A\cos(2\pi ft + \phi)$.

- ϕ is the phase constant and phase shifts the sine and cosine wave along the horizontal axis. Realize ϕ is not in the AP Physics 1 curriculum, however, it is very useful.
- For example: $x(t) = A\cos(2\pi ft) = A\sin\left(2\pi ft + \frac{\pi}{2}\right)$

Some useful points:

- θ was in radians in our derivation, therefore angles in the equations for simple harmonic motion are in radians and your calculator needs to be in radians when using these equations.
- ω is angular frequency which is ***not*** the same as frequency, f.
  - $\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} = 2\pi f$
- Yes, $T = \frac{2\pi}{\omega}$, is on the AP Physics equation sheets, however, you are much better served to remember and understand its derivation.

Flipping Physics Lecture Notes:

Simple Harmonic Motion - Velocity and Acceleration Equation Derivations

Previously[♥] we derived the equation on the AP Physics 1 equation sheet for an object moving in simple harmonic motion: $x\left(t\right)=A\cos\left(2\pi ft\right)$.

In order to derive the equations for velocity and acceleration, let's get position in terms of angular frequency: $f=\frac{1}{T}$ & $\omega=\frac{\Delta\theta}{\Delta t}=\frac{2\pi}{T}=2\pi f$ therefore $x\left(t\right)=A\cos\left(2\pi ft\right)=A\cos\left(\omega t\right)$.

Let's add the phase constant that shifts the wave along the horizontal axis: $x\left(t\right)=A\cos\left(\omega t+\phi\right)$.

Velocity is the derivative of position as a function of time. I know some of you might not have taken calculus yet and might not understand derivatives. Realize you need derivatives to derive velocity and acceleration simple harmonic motion equations. Some of this math might go over your heads, however, it is still useful to get some exposure to. ☺

Uses chain rule.

$$v=\frac{dx}{dt}=\frac{d}{dt}\left[A\cos\left(\omega t+\phi\right)\right]=A\frac{d}{dt}\left[\cos\left(\omega t+\phi\right)\right]=A\left[-\sin\left(\omega t+\phi\right)\right]\left[\frac{d}{dt}\left(\omega t+\phi\right)\right]$$

$$\Rightarrow v\left(t\right)=-A\sin\left(\omega t+\phi\right)\omega\Rightarrow v\left(t\right)=-A\omega\sin\left(\omega t+\phi\right)$$

Note: Because $-1\le\sin\theta\le1\rightarrow v_{max}=A\omega$

The derivation of acceleration is very similar:

Again, uses chain rule.

$$a=\frac{dv}{dt}=\frac{d}{dt}\left[-A\omega\sin\left(\omega t+\phi\right)\right]=-A\omega\frac{d}{dt}\left[\sin\left(\omega t+\phi\right)\right]=-A\omega\left[\cos\left(\omega t+\phi\right)\right]\left[\frac{d}{dt}\left(\omega t+\phi\right)\right]$$

$$\Rightarrow a=-A\omega\left[\cos\left(\omega t+\phi\right)\right]\left(\omega\right)\Rightarrow a\left(t\right)=-A\omega^2\cos\left(\omega t+\phi\right)$$

Again note: Because $-1\le\cos\theta\le1\rightarrow a_{max}=A\omega^2$

Remember: Because the derivation of these equations requires theta to be in radians, all angles in these equations need to be in radians and your calculator needs to be in radian mode when using equations for position, velocity, and acceleration as a function of time in simple harmonic motion.

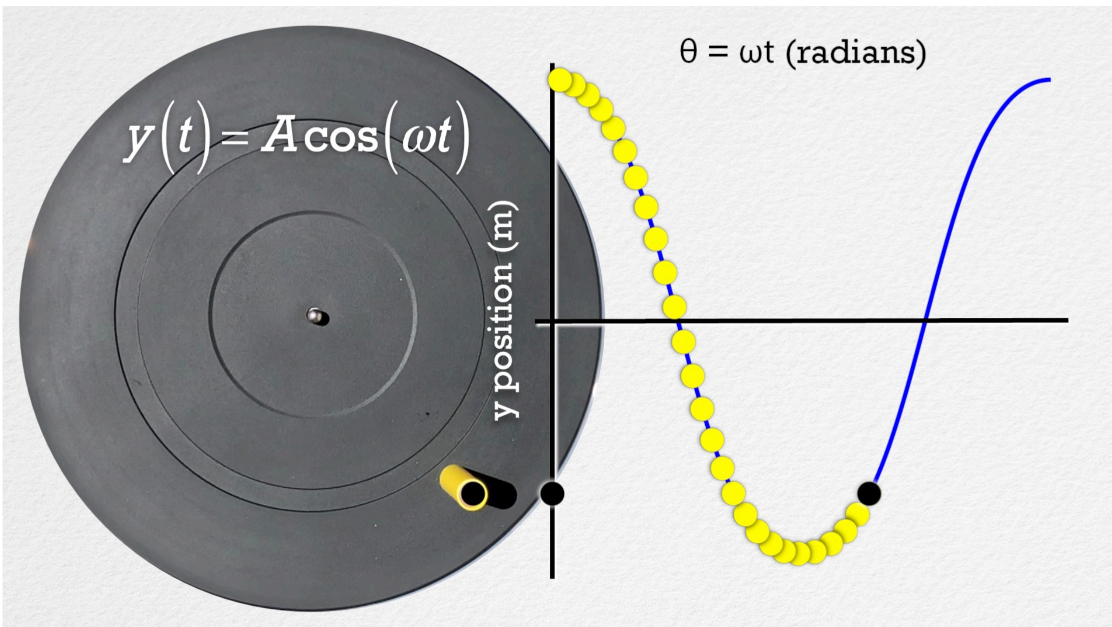
♥ https://www.flippingphysics.com/shm-position.html

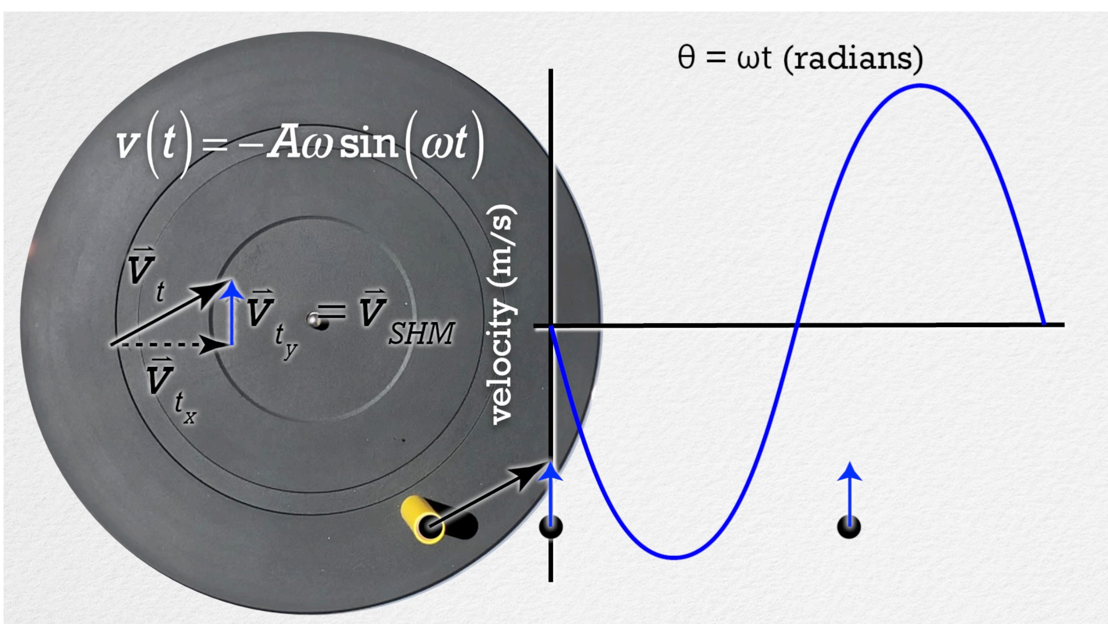
Simple Harmonic Motion – Graphs of Position, Velocity and Acceleration

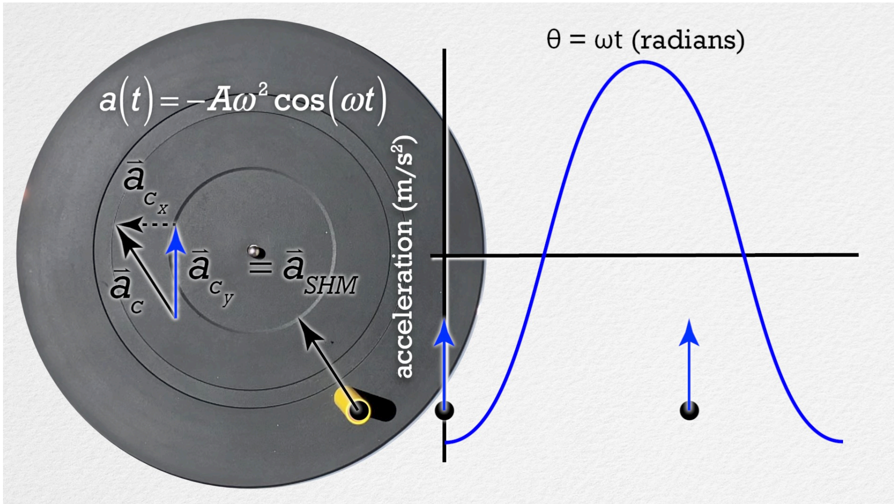
Previously we derived equations for position, velocity, and acceleration of an object in simple harmonic motion: $x(t)=A\cos(\omega t+\phi)$; $v(t)=-A\omega\sin(\omega t+\phi)$; $a(t)=-A\omega^2\cos(\omega t+\phi)$

Angular frequency, ω, derivation: $f=\frac{1}{T}$ & $\omega=\frac{\Delta\theta}{\Delta t}=\frac{2\pi}{T}=2\pi f$

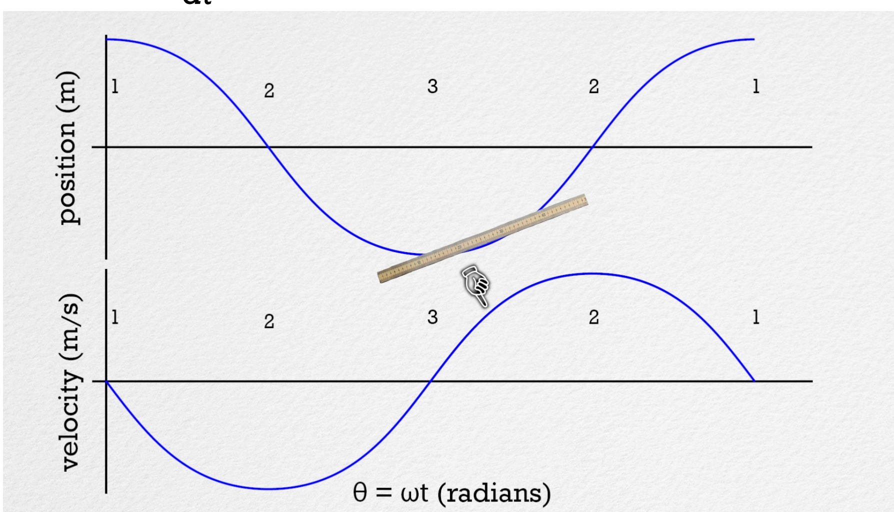
For our graphs, we are going to assume the phase constant, ϕ, is zero. In other words the graphs will not be phase shifted on the horizontal axis.



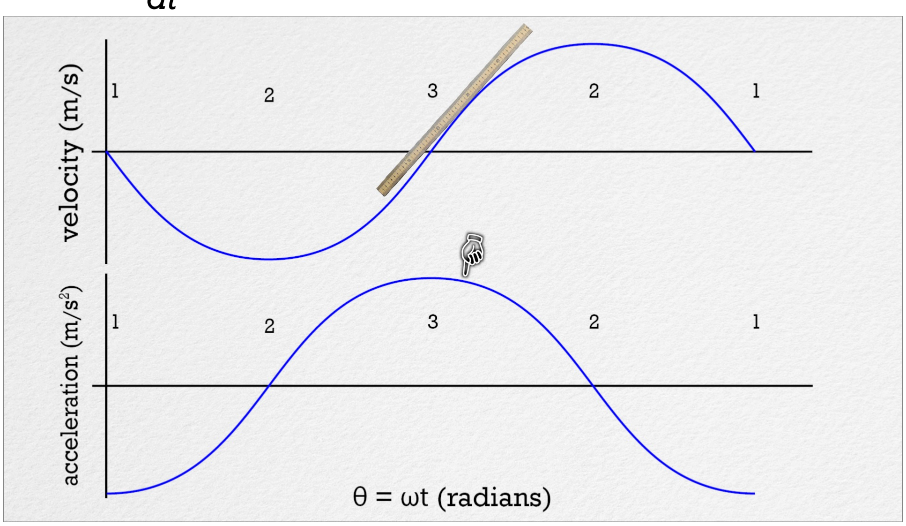




$$\vec{v} = \frac{d\vec{x}}{dt} \Rightarrow \text{velocity} = \text{slope of position vs. time}$$



$$\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow \text{acceleration} = \text{slope of velocity vs. time}$$

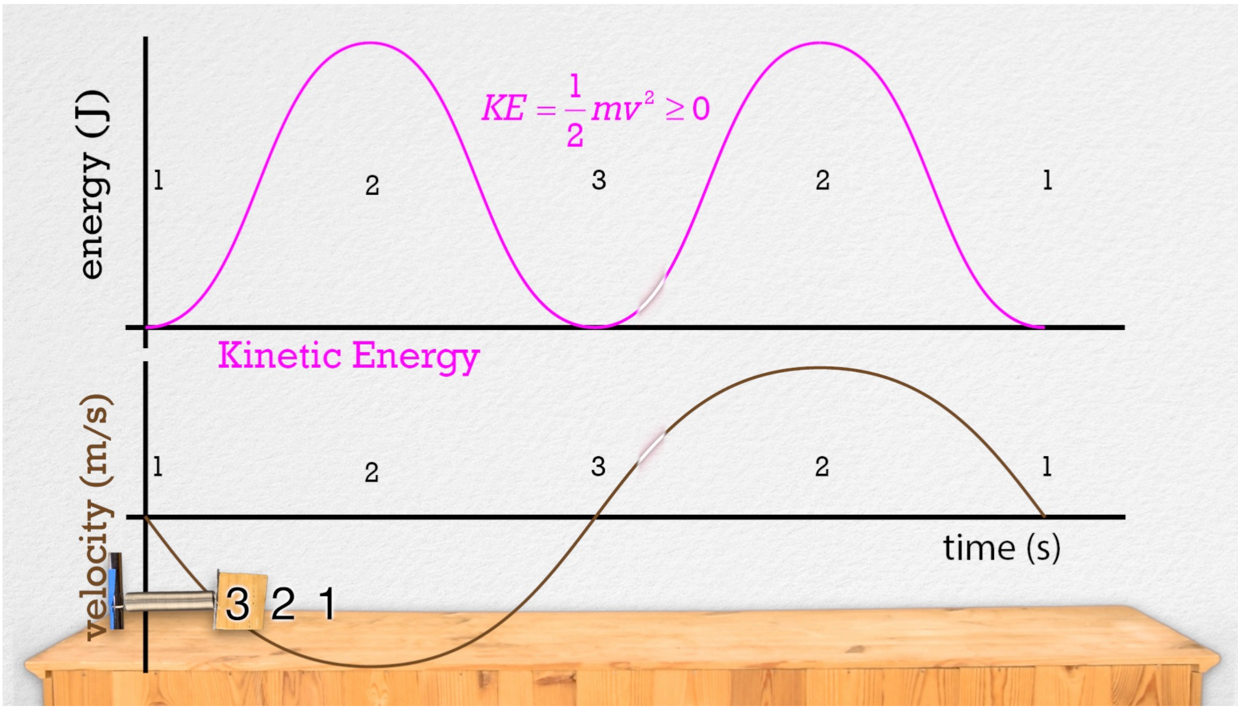


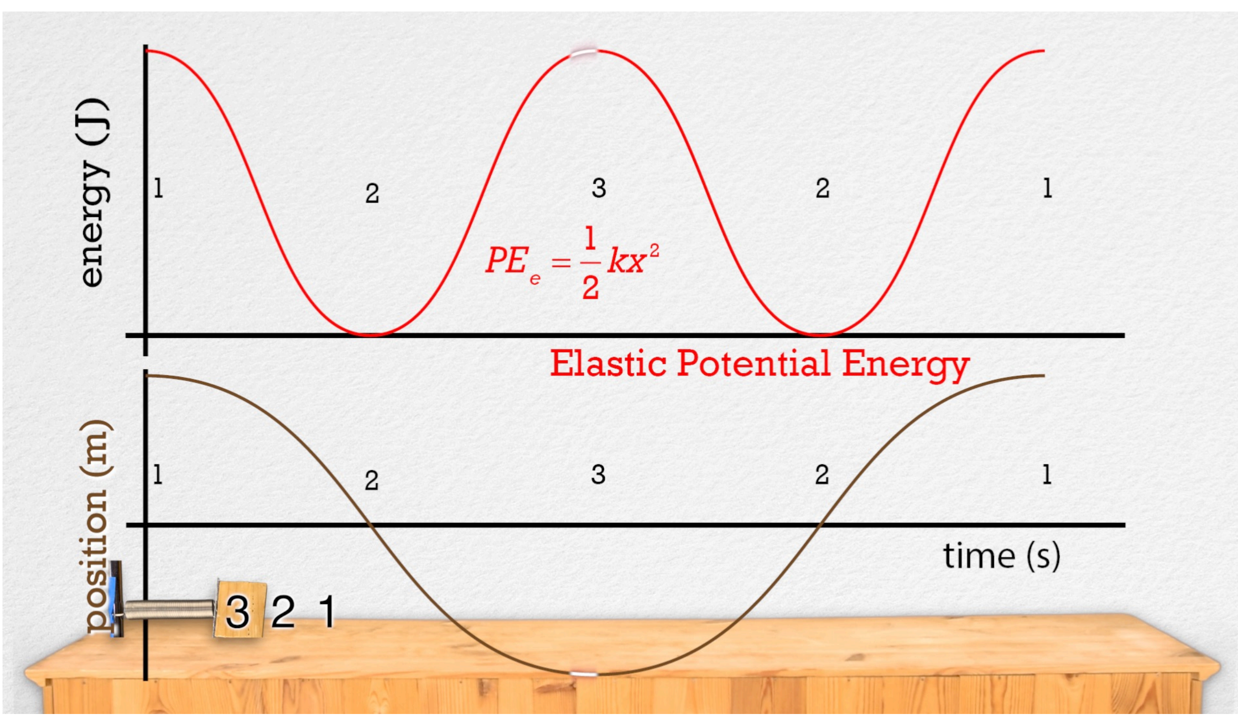
Flipping Physics Lecture Notes:

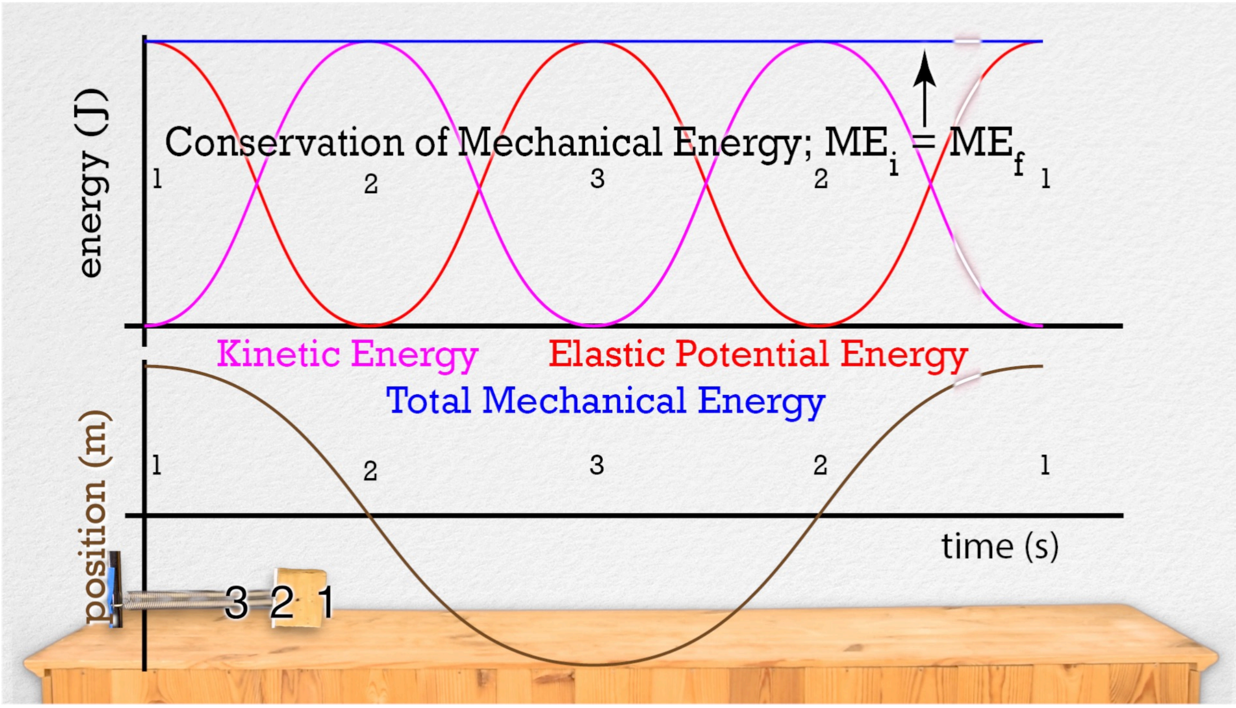
Simple Harmonic Motion – Graphs of Mechanical Energies

As previously determined, an object in simple harmonic motion has the following values:

Position(s)	x	Velocity	Acceleration
1 & 3	Maximum magnitude, A	0	Maximum magnitude, $A\omega^2$
2	0	Maximum magnitude, $A\omega$	0

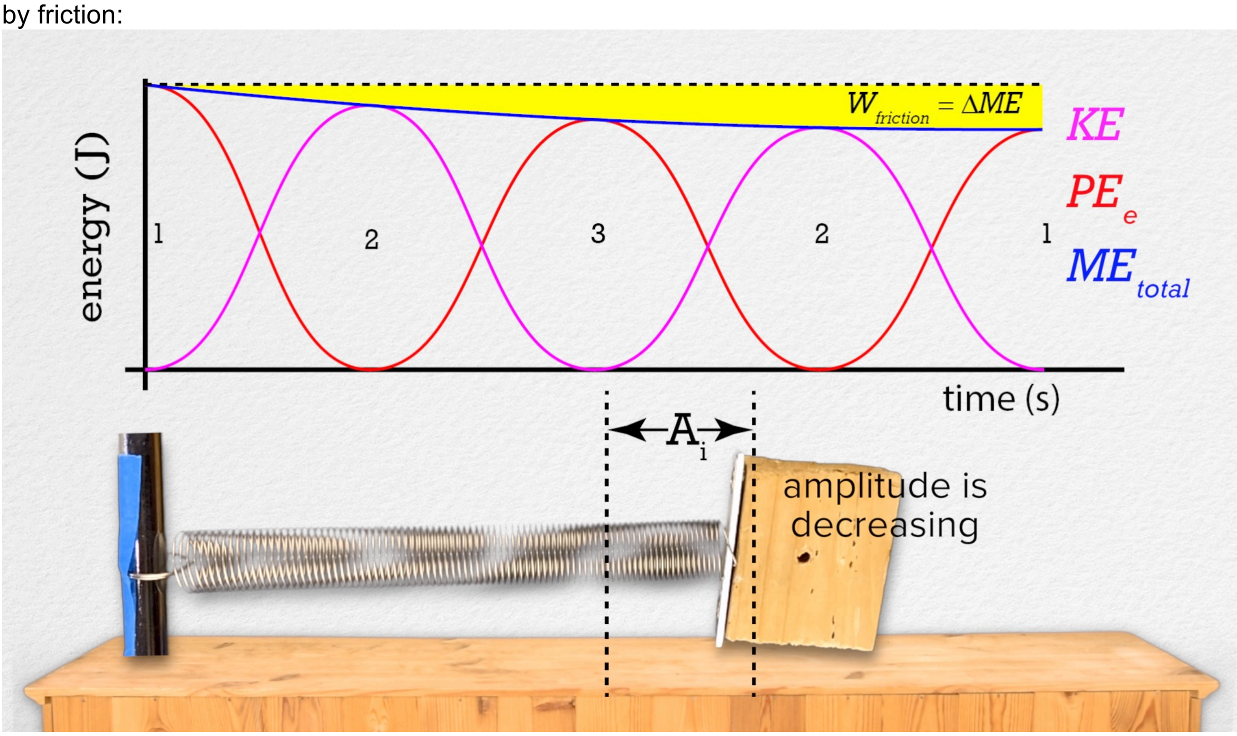






In the complete absence of friction, mechanical energy is conserved: $ME_{total} = \frac{1}{2}m\left(v_{\max}\right)^2 = \frac{1}{2}kA^2$

However, the reality is that some energy will be converted to internal energy of the spring via work done by friction:



Flipping Physics Lecture Notes:

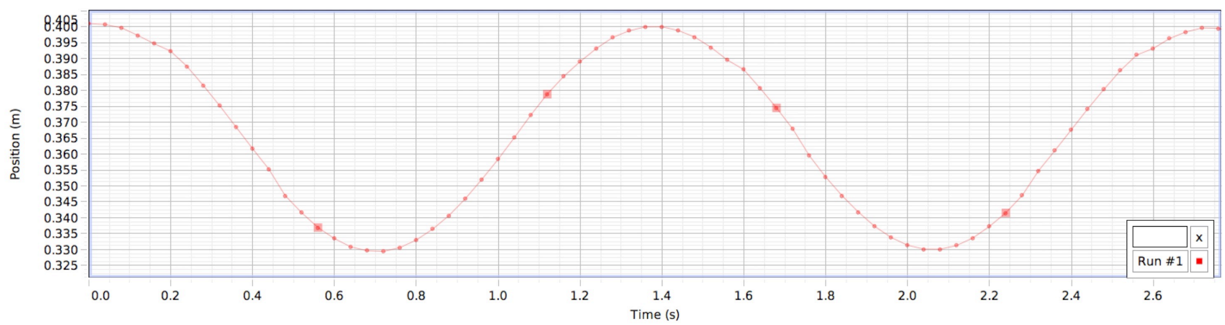
Demonstrating Position, Velocity, and Acceleration
of a Mass-Spring System

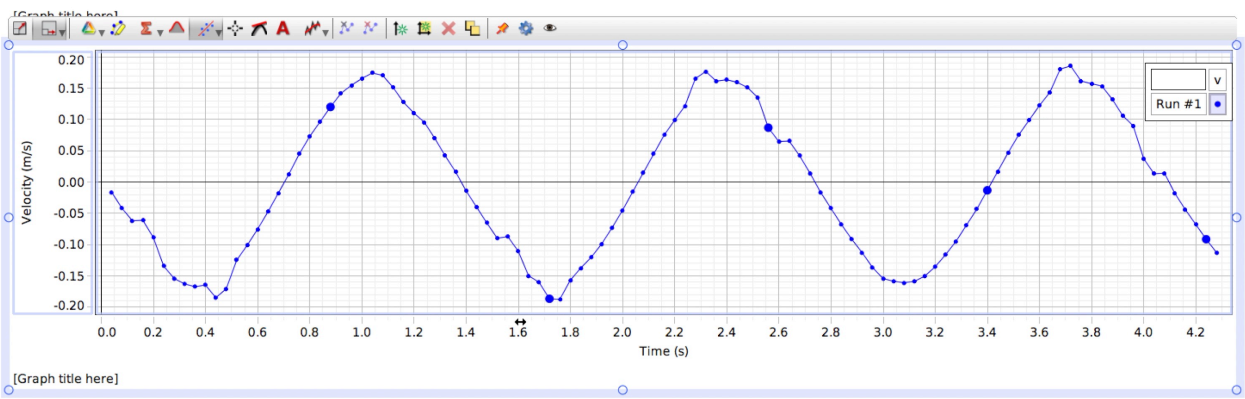
The basic equations of simple harmonic motion are:

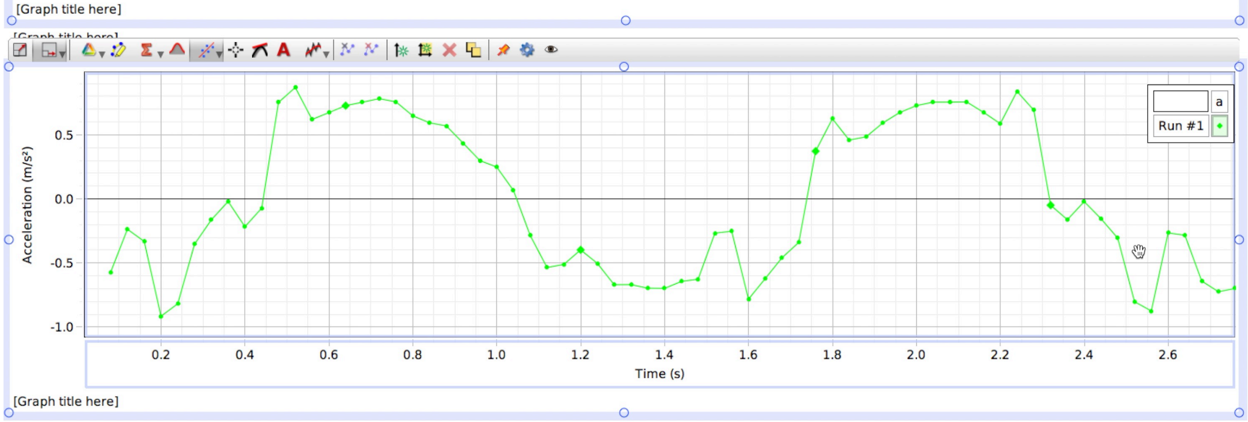
$$y(t)=A\cos(\omega t+\phi);\ v(t)=-A\omega\sin(\omega t+\phi);\ a(t)=-A\omega^2\cos(\omega t+\phi)$$

For our demonstrations we are going to assume the phase constant, ϕ, is zero. This means there is no phase shift in our demonstration.

The position, velocity, and acceleration as a function of time graphs:







Determining the period using the time for two cycles: $2T=2.72\text{sec}\Rightarrow T=\dfrac{2.72}{2}=1.36\text{sec}$

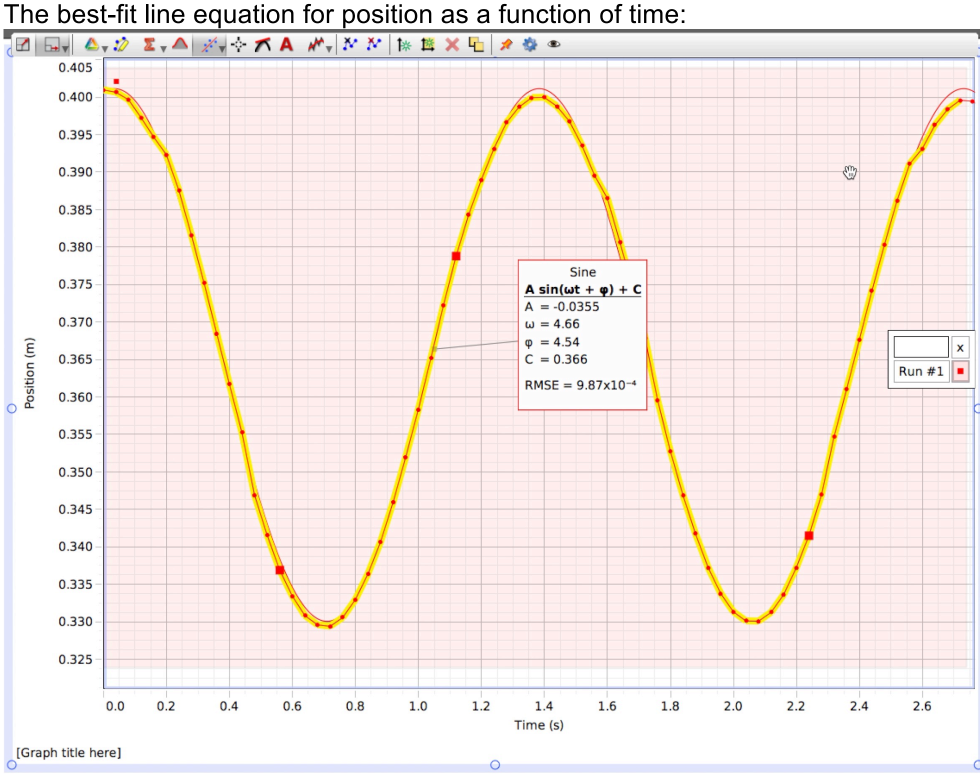
Determining the spring constant using the period and the mass:

$$T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow T^2 = 4\pi^2\left(\frac{m}{k}\right) \Rightarrow k = \frac{4\pi^2 m}{T^2} \Rightarrow k = \frac{4\pi^2(0.305)}{1.36^2} = 6.5100 \approx 6.51\frac{N}{m}$$

$$m = 305g \times \frac{1kg}{1000g} = 0.305kg$$

Just so you know, the 305 grams includes the mass of the mass hanging.

The best-fit line equation for position as a function of time:



[Table title here]

	Run #1 Position (m)	Time (s)
45	0.359	1.760
46	0.353	1.800
47	0.347	1.840
48	0.342	1.880
49	0.337	1.920
50	0.334	1.960
51	0.331	2.000
52	0.330	2.040
53	0.330	2.080
54	0.331	2.120
55	0.334	2.160
56	0.337	2.200
57	0.341	2.240
58	0.347	2.280
59	0.355	2.320
60	0.361	2.360
61	0.368	2.400
62	0.374	2.440
63	0.380	2.480
64	0.386	2.520
65	0.391	2.560
66	0.393	2.600
67	0.396	2.640
68	0.398	2.680
69	0.400	2.720
70	0.399	2.760
71	0.398	2.800
72	0.396	2.840
73	0.393	2.880

Determining period using angular frequency, ω:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{4.66} = 1.34832 \approx 1.35 \text{ sec}$$

Flipping Physics Lecture Notes:

Creating Circular Motion from Sine and Cosine Curves

Honestly, you've got to watch the video:

https://www.flippingphysics.com/circular-motion-sine-cosine.html

It's why I make them. You will understand this concept better if you watch the objects in simple harmonic motion creating the circular motion.