Flipping Physics Lecture Notes:

# Electric Charges and Electric Fields
## Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-electric-charges-electric-fields.html

Let's begin with Electric Charge:
- Electric charge is a fundamental property of all matter.
- Charge carriers are protons and electrons, where protons have a positive charge and electrons have a negative charge. Both protons and electrons have the same magnitude charge called the fundamental charge:
  - $e = 1.60 \times 10^{-19} C$ (The absolute value of the charge on a proton and electron.)
  - C stands for coulombs, the unit of charge. Note: Coulombs are not a base SI unit.
  - Charge is a scalar quantity.
- Charge is quantized. It comes in discrete quantities in intervals of the fundamental charge.
  - $Q = ne$
    - Q = net charge on the object
    - n = integer number of excess positive or negative charges on the object
      - n is positive for excess protons & n is negative for excess electrons
  - In other words, an object will never have a charge of +2.1 x $10^{-19}$ C because that is not an integer multiple of 1.60 x $10^{-19}$ C.
- The electric charge on an object is determined by the total number of charge carriers contained in the object.
  - The electric charge on an object with 4 protons and 6 neutrons is -3.62 x $10^{-19}$ C.
  - $Q_{\text{object}} = Q_{\text{protons}} + Q_{\text{electrons}} = (+4)(e) + (-6)(e) = (-2)(e)$
  - $\Rightarrow Q_{\text{object}} = (-2)\left(1.60 \times 10^{-19}\right) = -3.20 \times 10^{-19} C$
- Many objects we work with will be considered to be point charges (even if they are made up of many charge carriers)
  - A point charge is a model of a charge where the physical size of the charge (or charged system) is small enough to be considered to be negligible.

The Law of Charges states that:
- two charges with opposite signs attract one another
- two charges with the same sign repel one another

Coulomb's Law determines the electrostatic force between two charged objects:
- $\vec{F}_e = k\dfrac{(q_1)(q_2)}{r^2}\hat{r}$ or $\left|\vec{F}_e\right| = k\left|\dfrac{q_1 q_2}{r^2}\right|$ or $\left|\vec{F}_e\right| = \left(\dfrac{1}{4\pi\varepsilon_0}\right)\left|\dfrac{q_1 q_2}{r^2}\right|$ & $k = \dfrac{1}{4\pi\varepsilon_0}$
  - $F_e$ is the electrostatic force on each charged particle.
  - k is the Coulomb constant and it equals 8.99 x $10^9$ N • m / $C^2$.
  - $\varepsilon_0$ is the permittivity of free space or vacuum permittivity; it equals 8.85 x $10^{-12}$ $C^2$ / N • $m^2$.
    - We will better define permittivity when we get to capacitors and dielectrics.
  - $q_1$ and $q_2$ are the charges on the two interacting charged particles.
  - r is the distance between the centers of charge of the two charges
  - $\vec{F}_e = k\dfrac{(q_1)(q_2)}{r^2}\hat{r}$ is the vector version.
    - If the two charges have the same sign, the force is positive and repulsive.
    - If the two charges have opposite signs, the force is negative and attractive.
  - $\left|\vec{F}_e\right| = k\left|\dfrac{q_1 q_2}{r^2}\right|$ is the scalar version and has no direction.
  - According to AP Physics C: Electricity and Magnetism guidelines, you will only be expected to calculate electric forces between four or fewer charged objects or systems. However, you may be required to analyze more than four charged objects in situations which have high symmetry.

The reality is that electrostatic forces, which can be determined using Coulomb's Law, cause many of the forces which we work with on a macroscopic level. For example, force normal, forces of static and kinetic friction, and the force of tension. There are just too many microscopic electrostatic forces to reasonably calculate, and we model the net force caused by all of these microscopic electrostatic forces as these macroscopic forces.

Note that Coulomb's Law is similar to Newton's Universal Law of Gravitation:

- Coulomb's Law: $\left|\vec{F}_e\right| = k\left|\frac{q_1 q_2}{r^2}\right|$ and the Universal Law of Gravitation: $\left|\vec{F}_g\right| = \frac{Gm_1 m_2}{r^2}$
- r is distance between centers of charge in the case of Coulomb's law, or distance between centers of mass in the case of the Universal Law of Gravitation.
- Electrostatic forces can be attractive or repulsive, however, gravitational forces can only be attractive.
- Notice the difference between the constants in the equations:
    - $k = 8.99 \times 10^9 \frac{N \cdot m}{C^2}$ & $G = 6.67x10^{-11} \frac{N \cdot m^2}{kg^2}$
    - The Coulomb constant is much larger than the universal gravitational constant
    - When two objects have both mass and electric charge, most often the electric force is so much larger than the gravitational force that the gravitational force is negligible.
        - However, when we are considering large scales like people, planes, pineapples, planets, and Prii[1], gravitational forces are large enough to make electric forces negligible because large scale objects are usually nearly electrically neutral. In other words, their total number of protons and electrons are nearly equal.

The Law of Conservation of Charge:

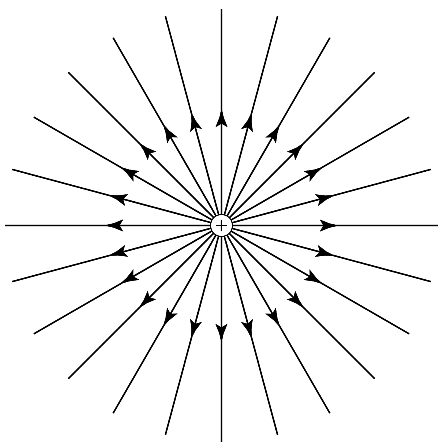
- The net charge of an isolated system does not change.
    - Typically, electrons are the charges which move in a system and if there is no way for electrons to leave or enter the system (an isolated system), then the net charge of the system will not change.
- Charging by friction involves rubbing one object on another object (fur on a balloon, for example). When this happens, electrons move from the fur to the balloon.
    - The balloon gains negative electrons and the net charge on the balloon decreases.
    - The fur loses electrons and the net charge on the fur increases.
    - The net charge on the balloon-fur system remains the same because the total number of protons and electrons in the system remains the same.
- Bringing two objects close to one another will cause electrostatic forces between charges and change the distribution of the electrons in the objects. This can polarize the objects, however, the net charge on each object will remain the same if electrons are not allowed to flow between the objects.
- When a charged object is brought close to a neutral object, charge-induced separation of charges within the neutral object can occur.
- The only way to change the charge of a system is to add or remove charged particles from the system. In the matter most familiar to us, the electrons are the charged particles that can be added or removed from the system to change its charge. In this case the system is *not* isolated, and charge is *not* conserved.
- A "ground" has, relatively speaking, an infinite number of electrons which we can pull from it, or we can give to it. We can "ground" a system by electrically connecting it to the Earth which is a "ground" or a neutrally charged, infinite well of electrons. When we ground a system, the system is *not* isolated, and charge is *not* conserved.

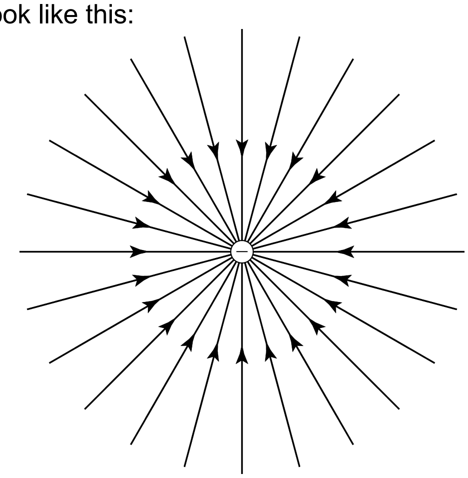
---

[1] https://pressroom.toyota.com/toyota-announces-the-plural-of-prius/

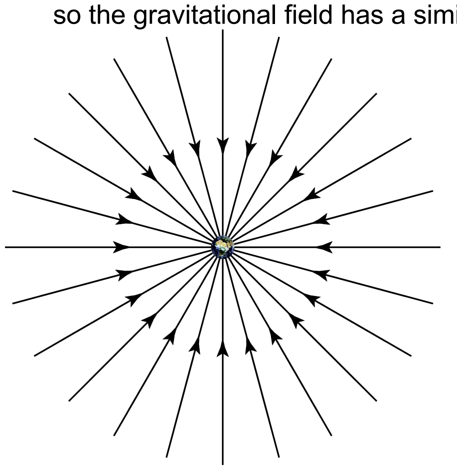
Electric Fields:

- If we were to place a positive test charge in an electric field, it would experience an electrostatic force. An electric field is the ratio of the electrostatic force the test charge would experience and the charge of the test charge.
    - A positive test charge is a charge which is small enough not to measurably change the electric field it is placed in. Electric field directions are defined according to the direction of the net electrostatic force on a positively charged test charge.
- The equation for an electric field and its units are: $\vec{E} = \frac{\vec{F}_e}{q} \Rightarrow \frac{N}{C}$
- Notice the electric field and the electrostatic force experienced by a positive charge in the electric field will be in the same direction. (Both electric field and electrostatic force are vectors in the equation.)
- That means the electric field which surrounds a point charge is: $\vec{E} = \frac{\vec{F}_e}{q} \Rightarrow E_{\text{point charge}} = \frac{\frac{kqQ}{r^2}}{q} = \frac{kQ}{r^2}$
- And the electric field maps for **isolated** point charges look like this:

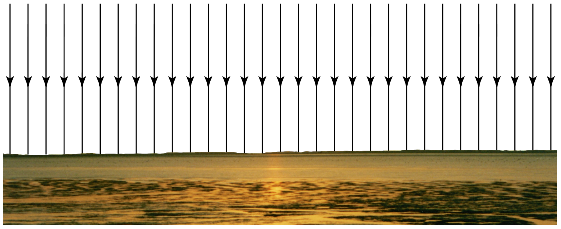
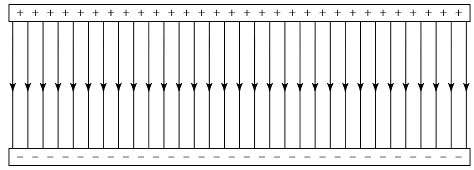


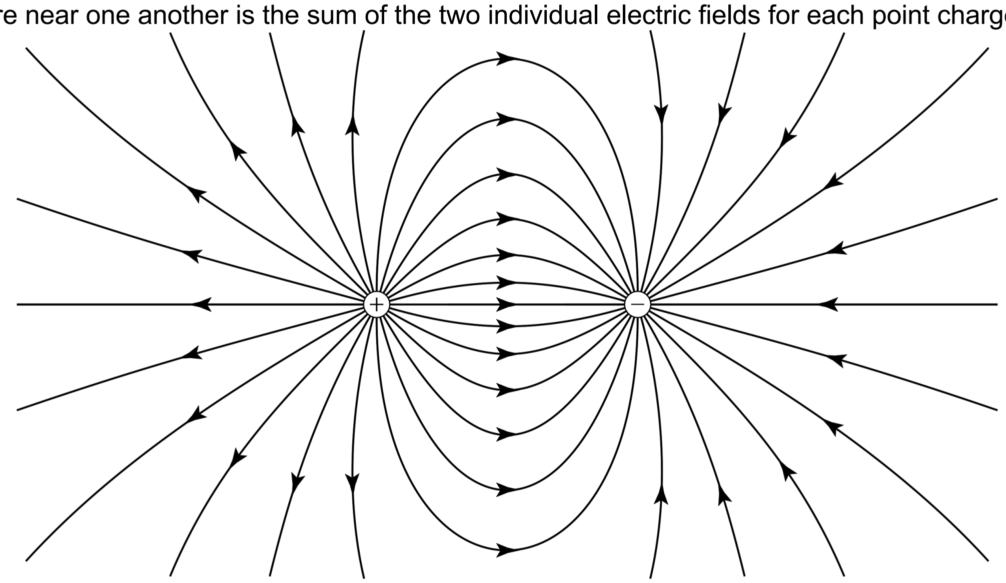

- Notice the similarity to the gravitational field around a planet. The equation has a similar format, so the gravitational field has a similar format: $g = \frac{F_g}{m} \Rightarrow g_{\text{point mass}} = \frac{\frac{GmM}{r^2}}{m} = \frac{GM}{r^2}$



- And the electric field which exists between two large, parallel, oppositely charged plates is similar to the gravitational field close to the surface of a planet:

-
- Remember electric field is a vector which means that the electric field for two point charges which are near one another is the sum of the two individual electric fields for each point charge.



- Electric field maps like the one above are simplified models and vector maps which show the magnitude and direction of the electric field for the entire region.
- Electric Field Lines Basics:
  - In the direction a small, positive, test charge would experience an electrostatic force
  - Electric field lines per unit area is proportional to electric field strength
    - Higher density electric field lines = higher electric field strength
  - Start on a positive charge and end on a negative charge
    - or infinitely far away if more of one charge than another
  - Always start perpendicular to the surface of the charge
  - Electric field lines never cross

Conductors vs. Insulators:
- *Conductors* allow electrons to move rather easily. This is because conductors have electrons which are loosely bound to their atoms which allows electrons to flow.
  - Examples: aluminum, stainless-steel, and gold.
- *Insulators* resist the motion of electrons. Insulators have electrons which are tightly bound to their atoms which does not allow electrons to flow.
  - Examples: plastic, rubber, glass, and paper.
- *Semiconductors* are materials which are somewhere in between conductors and insulators. Electrons have some resistance to flow, and the precise resistance to flow is controlled by the composition of the materials.
  - Examples: silicon, germanium, and gallium arsenide.
  - Example uses: diodes, transistors, amplifiers, solar cells, and light emitting diodes (LEDs)

Flipping Physics Lecture Notes:
Continuous Charge Distributions
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-continuous-charge-distributions.html

Continuous Charge Distribution: A charge that is not a point charge. In other words, a charge which has shape and continuous charge distributed throughout the object.

In order to find the electric field which exists around a continuous charge distribution, we can use Coulomb's Law and the equation definition of an electric field. We consider the charged object to be made up of an infinite number of infinitesimally small point charges dq and add up the infinite number of electric fields via superposition. It's an integral.

$$\vec{F}_e = k\frac{(q_1)(q_2)}{r^2}\hat{r} \ \& \ \vec{E} = \frac{\vec{F}_e}{q} \Rightarrow \vec{E}_{\text{point charge}} = k\frac{\frac{(q)(Q)}{r^2}}{q}\hat{r} = \frac{kQ}{r^2}\hat{r}$$

$$\Rightarrow d\vec{E} = \frac{k(\mathrm{d}q)}{r^2}\hat{r} \Rightarrow \int d\vec{E} = \int \frac{k(\mathrm{d}q)}{r^2}\hat{r} \Rightarrow \vec{E}_{\text{continuous charge distribution}} = k\int\frac{\mathrm{d}q}{r^2}\hat{r}$$

Realize that, for AP Physics C: Electricity and Magnetism, students are only expected to be able to use this equation to determine electric fields around continuous charge distributions with high symmetry. The specific examples students are responsible for are:

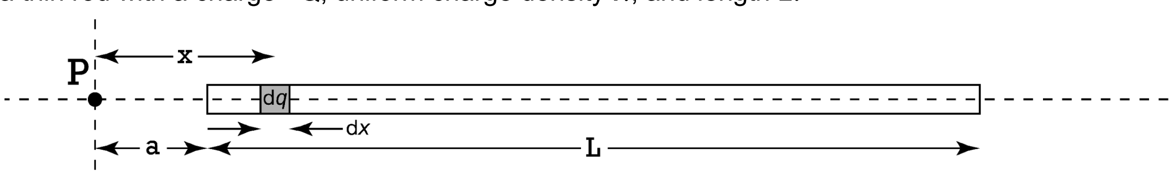
- An infinitely long, uniformly charged wire or cylinder at a distance from its central axis
- A thin ring of charge at a location along the axis of the ring
- A semicircular arc or part of a semicircular arc at its center
- A finite wire or line of charges at a distance that is collinear with the line of charge or at a location along its perpendicular bisector.

Quick review of charge densities:

linear charge density, $\lambda = \frac{Q}{L}$ in $\frac{C}{m}$ & surface charge density, $\sigma = \frac{Q}{A}$ in $\frac{C}{m^2}$

& volumetric charge density, $\rho = \frac{Q}{V}$ in $\frac{C}{m^3}$

Let's do an example. Let's determine the electric field at point P, which is located a distance "a" to the left of a thin rod with a charge +Q, uniform charge density $\lambda$, and length L.




Notice that, if we were to place a positive point charge at point P, it would experience a force to the left from every dq or every infinitesimally small part of the wire. Therefore, we already know the direction of the electric field at point P, it will be to the left or in the negative "i" direction. Now let's solve for the electric field:

$$\vec{E} = k\int\frac{\mathrm{d}q}{r^2}\hat{r} \Rightarrow \vec{E} = k\int\frac{\mathrm{d}q}{x^2}\left(-\hat{\imath}\right) \ \& \ \lambda = \frac{Q}{L} = \frac{\mathrm{d}q}{\mathrm{d}x} \Rightarrow \mathrm{d}q = \lambda\mathrm{d}x \ \& \ Q = \lambda L$$

$$\Rightarrow \vec{E} = -k\hat{\imath}\int\frac{\lambda\mathrm{d}x}{x^2} = -k\lambda\hat{\imath}\int_a^{a+L}\left(\frac{1}{x^2}\right)\mathrm{d}x = -k\lambda\hat{\imath}\int_a^{a+L}\left(x^{-2}\right)\mathrm{d}x$$

$$\Rightarrow \vec{E} = -k\lambda\hat{\imath}\left[\frac{x^{-1}}{-1}\right]_a^{a+L} = k\lambda\hat{\imath}\left[\frac{1}{x}\right]_a^{a+L} = k\lambda\hat{\imath}\left[\frac{1}{a+L} - \frac{1}{a}\right] = k\lambda\hat{\imath}\left[\frac{a-(a+L)}{a(a+L)}\right]$$

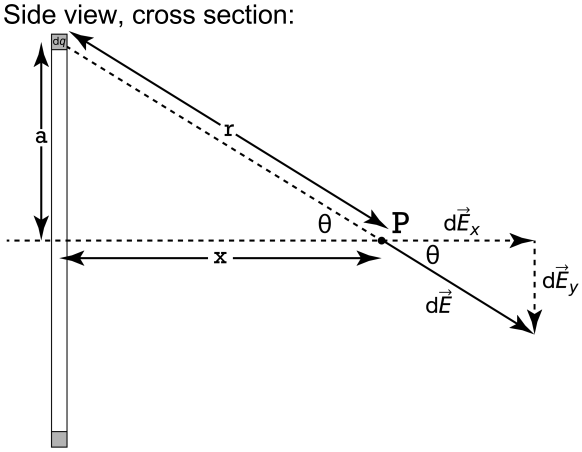
$$\Rightarrow \vec{E} = k\lambda\hat{\imath}\left[\frac{a-a-L}{a(a+L)}\right] = -\frac{k\lambda L}{a(a+L)}\hat{\imath} \Rightarrow \vec{E} = -\frac{kQ}{a(a+L)}\hat{\imath}$$

if $a >> L$ then $a + L \approx a$ & $\vec{E} = -\frac{kQ}{a^2}\hat{\imath}$

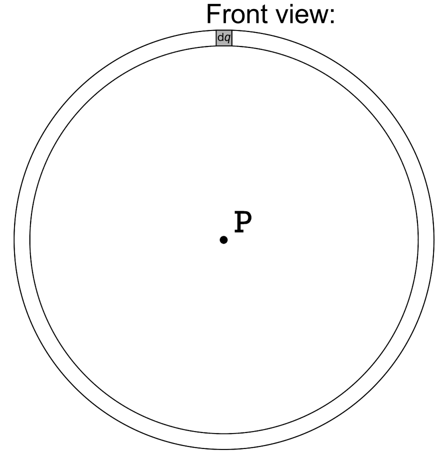
In other words, if we get far enough from the charged rod, it acts like a point charge. ☺

And, because they are so fun, another example! Let's determine the electric field caused by a uniformly charged ring of charge +Q, with radius a, at point P which is located on the axis of the ring a distance x from the center of the ring.

Side view, cross section:




Front view:




Knowns: a, x, Q

$$\vec{E} = k\int\frac{dq}{r^2}\hat{r} \Rightarrow d\vec{E} = k\left(\frac{dq}{r^2}\right)\hat{r} = k\left(\frac{dq}{a^2+x^2}\right)\hat{r}$$

However, all dE's in the vertical plane cancel out because there is an equal but opposite component of dE caused by the dq on the opposite side of the ring. In the figure that is $dE_y$.

$$d\vec{E}_x = d\vec{E}\cos\theta = k\left(\frac{dq}{a^2+x^2}\right)\hat{\imath}\cos\theta \Rightarrow \vec{E}_P = \int\left(\frac{k}{a^2+x^2}\hat{\imath}\cos\theta\right)dq$$

$$\& \quad \cos\theta = \frac{A}{H} = \frac{x}{r} = \frac{x}{\sqrt{a^2+x^2}} \Rightarrow \vec{E}_P = \int\left[\left(\frac{k}{a^2+x^2}\right)\left(\frac{x}{\sqrt{a^2+x^2}}\right)\hat{\imath}\right]dq$$

Note: $(a^2+x^2)(a^2+x^2)^{\frac{1}{2}} = (a^2+x^2)^{\frac{2}{2}}(a^2+x^2)^{\frac{1}{2}} = (a^2+x^2)^{\frac{3}{2}}$

$$\Rightarrow \vec{E}_P = \int\left(\frac{kx}{(a^2+x^2)^{\frac{3}{2}}}\right)\hat{\imath}\,dq = \left(\frac{kx}{(a^2+x^2)^{\frac{3}{2}}}\right)\hat{\imath}\int dq \Rightarrow \vec{E}_P = \left(\frac{kQx}{(a^2+x^2)^{\frac{3}{2}}}\right)\hat{\imath}$$

Note:

$$\text{if } x >> a \text{ then } a^2 + x^2 \approx x^2 \;\&\; \vec{E}_P \approx \left(\frac{kQx}{\left(x^2\right)^{\frac{3}{2}}}\right)\hat{\imath} = \left(\frac{kQx}{x^3}\right)\hat{\imath} \qquad \Rightarrow \vec{E}_P \approx \left(\frac{kQ}{x^2}\right)\hat{\imath}$$

That's right, if you get far enough from the uniformly charged ring, it acts like a point particle!
If that sounds familiar, that is because this is true for all continuous charge distributions. If you get far enough away from them, that their own size is small by comparison to the distance, they all have electric fields which are similar to point particles.

$$\text{if } x << a \text{ then } a^2 + x^2 \approx a^2 \;\&\; \vec{E}_P \approx \left(\frac{kQx}{\left(a^2\right)^{\frac{3}{2}}}\right)\hat{\imath} \Rightarrow \vec{E}_P \approx \left(\frac{kQx}{a^3}\right)\hat{\imath}$$

Note:
And if we use a negative charge, then the force is to the left or towards the center of the ring:

$$\sum F_x = -F_e = ma_x \Rightarrow -qE = -q\left(\frac{kQx}{a^3}\right) = ma_x \Rightarrow a_x = -\left(\frac{kqQ}{ma^3}\right)x$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{kqQ}{ma^3}\right)x \;\&\; \frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \omega = \sqrt{\frac{kqQ}{ma^3}} = \frac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\frac{ma^3}{kqQ}}$$

$$x(t) = A\cos(\omega t + \phi) \Rightarrow x_{max} = A$$

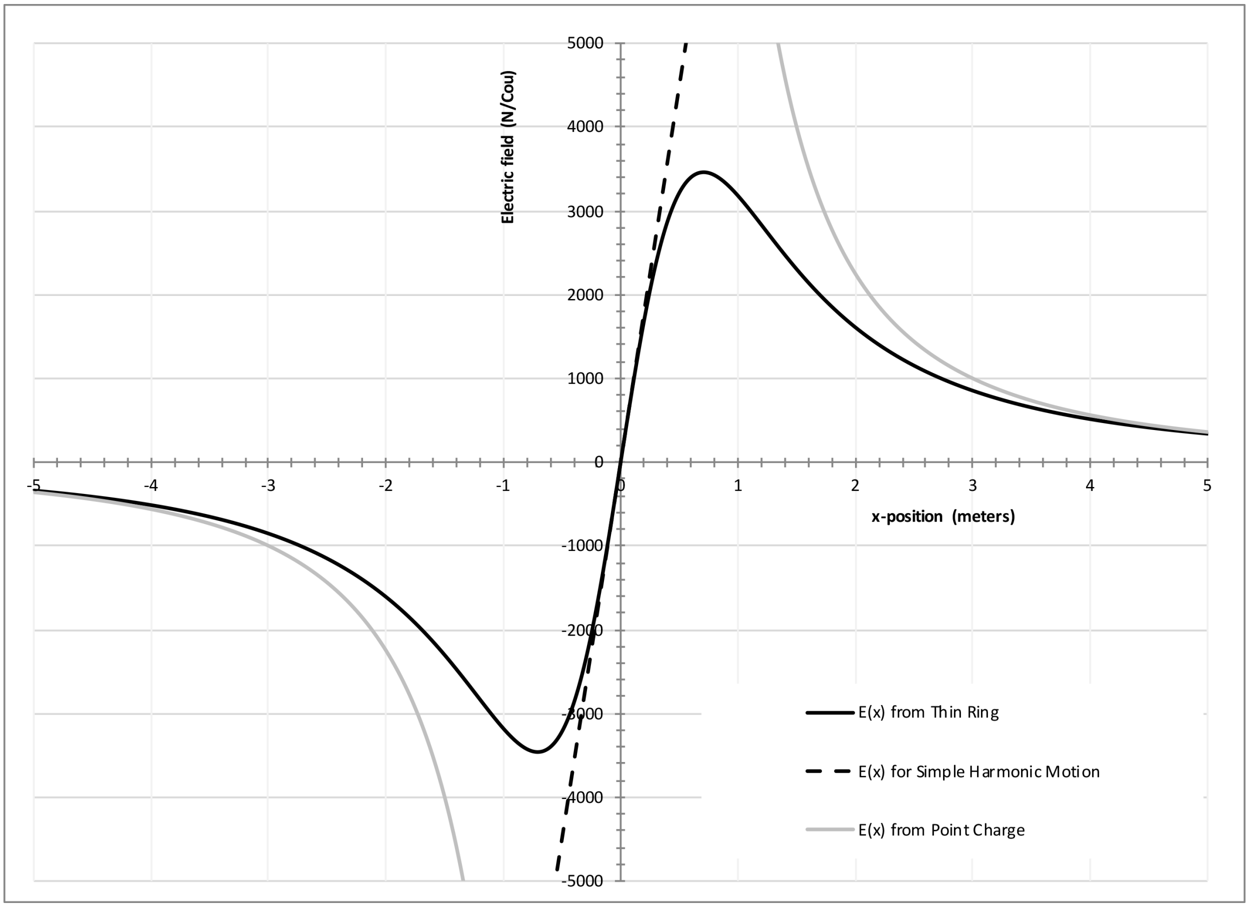
$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi) \Rightarrow v_{max} = A\omega$$

$$a(t) = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \phi) \Rightarrow a_{max} = A\omega^2$$

That's right, the negative charge will move in simple harmonic motion about the center of the ring.

A bonus graph from Carl Hansen: (Thank you Carl Hansen!)

This following graph shows the equation we derived for the electric field along the axis of a thin ring of a uniform charge distribution, of radius a = 1m, and charge Q = +1μC. The plot shows the linear profile needed for simple harmonic motion close to the origin, and the inverse square law far away. The maximum electric field occurs around x = 0.7m, as a turning point to transition between the two trends.

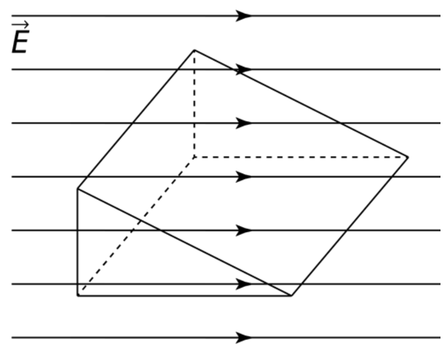


Flipping Physics Lecture Notes:
Electric Flux and Gauss' Law
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-electric-flux-gauss-law.html

Flux is defined as any effect that appears to pass or travel through a surface or substance, however, realize that effect does not need to move. Hence, "appears to".

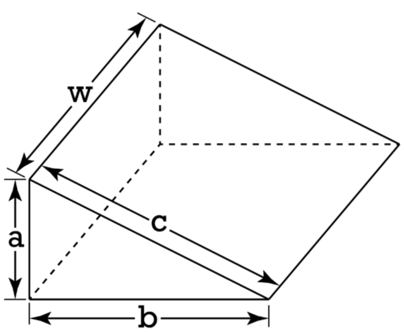
Electric flux is the measure of the amount of electric field which passes through a defined area. The equation for electric flux of a uniform electric field is:

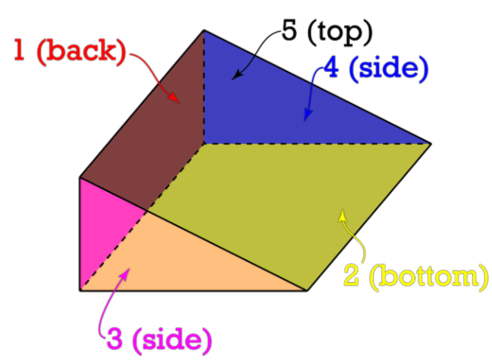
- $\Phi_E = \vec{E} \cdot \vec{A} = EAcos\theta$
- $\Phi$ is the uppercase, Greek letter phi
- E is the uniform electric field *(use magnitude)*
- A is the area of the surface through which the uniform electric field is passing *(use magnitude)*
- θ is the angle between the directions of E and A
    - Notice this is the same form as the equation for work. This means you use the magnitudes of E and A, and cosθ determines if the electric flux is positive or negative
    - $W = \vec{F} \cdot \Delta\vec{r} = F\Delta r \cos\theta$
- Electric flux is a scalar
- The units for electric flux are $\frac{N \cdot m^2}{C}$



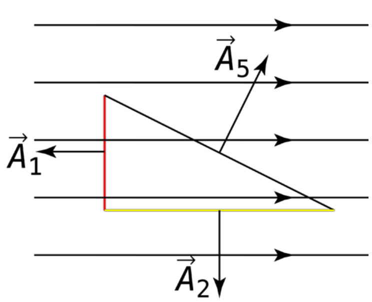

Usually, electric flux is through some sort of closed surface. So, let's do an example and determine the net electric flux of a uniform, horizontal electric field through a right triangular box.

Let's define and label the dimensions and sides of the triangular box as:




And now we can determine the electric flux through each side:




Electric flux for Area 1 (back): $\theta_1$ is 180° because Area 1 is to the left or out of the rectangular box and the electric field is to the right.

$$\Phi_1 = EA_1cos\theta_1 = E(aw)\cos(180°) = -Eaw$$

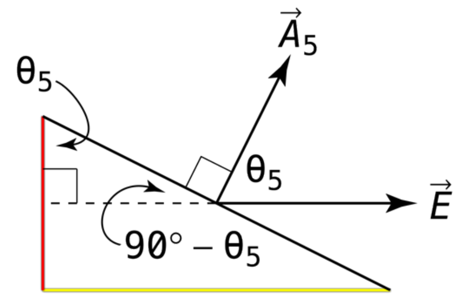
Electric flux for Area 2 (bottom): $\theta_2$ is 90° because Area 2 is down or out of the rectangular box and the electric field is to the right.

$$\Phi_2 = EA_2cos\theta_2 = E(bw)\cos(90°) = 0$$

Electric flux for Areas 3 and 4 (sides): $\theta_3$ and $\theta_4$ are both 90° because Area 3 is out of the page and Area 4 is into the page (and the electric field is to the right).

$$\Phi_3 = EA_3 cos\theta_3 = E\left(\frac{1}{2}ba\right)\cos(90^\circ) = 0 = \Phi_4$$

Electric flux for Area 5 (top): To understand why $\cos\theta_5 = a/c$, we need to draw another diagram.




$$\cos\theta_5 = \frac{A}{H} = \frac{a}{c}$$

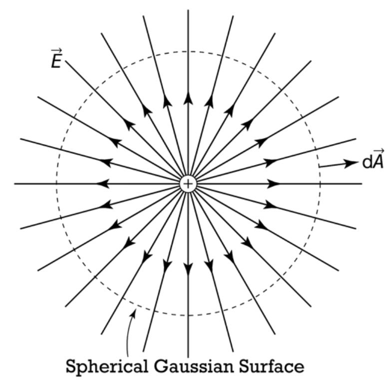
$$\Phi_5 = EA_5 cos\theta_5 = E(cw)\left(\frac{a}{c}\right) = Eaw$$

And the total electric flux through the entire triangular box is:

$$\Phi_{\text{total}} = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 = -Eaw + 0 + 0 + 0 + Eaw = 0$$

Notice that:

- When an electric field is going into a closed surface, the electric flux is negative.
- When an electric field is coming out of a closed surface, the electric flux is positive.




Let's now do another example. Let's determine the electric flux passing through a sphere which is concentric to and surrounds a positive point charge.

Notice we cannot use $\Phi_E = \vec{E}\cdot\vec{A} = EAcos\theta$ because the electric field is not uniform. We need to use the integral equation for electric flux:

$$\Phi_E = \vec{E}\cdot\vec{A} \Rightarrow d\Phi_E = \vec{E}\cdot d\vec{A} \Rightarrow \int d\Phi_E = \int \vec{E}\cdot d\vec{A} \Rightarrow \Phi_E = \int \vec{E}\cdot d\vec{A}$$

$$\Phi_E = \int \vec{E}\cdot d\vec{A} = \int E\,dA\cos\theta = \int E\,dA\cos(0) = \int E\,dA = E\int dA = EA$$

$$\vec{F}_{21} = k\frac{(q_1)(q_2)}{r^2}\hat{r}_{21} \;\&\; \vec{E} = \frac{\vec{F}_e}{q} \Rightarrow E_{+\text{ point charge}} = \frac{\frac{kqQ}{r^2}}{q} = \frac{kQ}{r^2} \;\&\; A_{\text{sphere}} = 4\pi r^2$$

$$\Rightarrow \Phi_E = \left(\frac{kQ}{r^2}\right)\left(4\pi r^2\right) = 4\pi kQ = 4\pi\left(\frac{1}{4\pi\varepsilon_0}\right)Q = \frac{Q}{\varepsilon_0}$$

In other words, the electric flux through a closed Gaussian surface is:

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

This is Gauss' law!

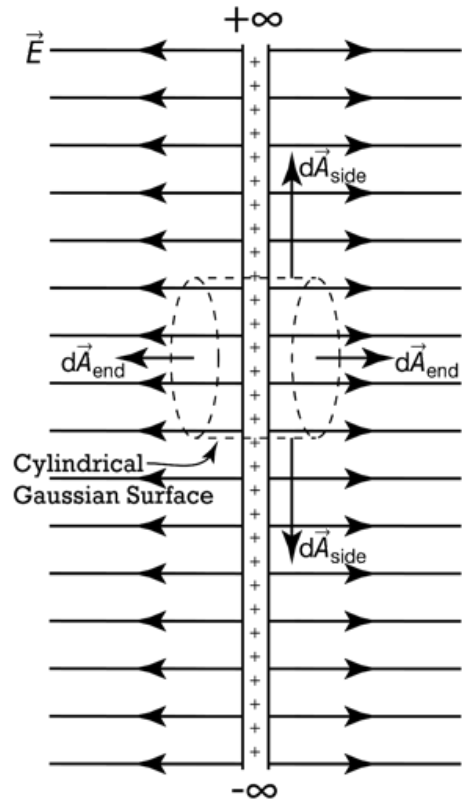
Gauss' law relates electric flux through a Gaussian surface to the charge enclosed by the Gaussian surface:

- A Gaussian surface is a three-dimensional closed surface
- While the Gaussian surfaces we usually work with are imaginary, the Gaussian surface could actually be a real, physical surface
- Typically, we choose the shapes of our Gaussian surfaces such that the electric field generated by the enclosed charge is either perpendicular or parallel to the various sides of the Gaussian surface. This greatly simplifies the surface integral because all the angles are multiples of 90° and the cosine of those angles have a value of -1, 0, or 1.
- As long as the amount of charge enclosed in a Gaussian surface is constant, the total electric flux through the Gaussian surface does not depend on the size of the Gaussian surface.
- Gauss' law is the first of Maxwell's equations which are a collection of equations which fully describe electromagnetism.

Notice then that, if the net charge inside a closed Gaussian surface is zero, then the net electric flux through the Gaussian surface is zero.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0} = \frac{0}{\varepsilon_0} = 0$$

This is why the net electric flux through the closed rectangular box in our first example was zero.




Let's do another example using Gauss' law. Determine the electric field which surrounds an infinitely large, thin plane of positive charges with uniform surface charge density, σ:

First off, we know the electric field will be directed normal to and away from the infinite plane of positive charges. This is because the plane is infinitely large; therefore, every component of the electric field, dE, which is parallel to the plane of charges and is caused by infinitesimally small, charged pieces of the plane, dq, will cancel out leaving only electric field components of dE which are perpendicular to the plane and directed away from the plane.

We pick a Gaussian surface such that it is a cylinder with ends parallel to the plane of charges and a side parallel to the electric field and use Gauss' law. The two ends of the Gaussian cylinder are equidistant from the charged plane.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\Phi_E = \int_{\text{side}} \vec{E} \cdot d\vec{A}_{\text{side}} + \int_{\substack{\text{left}\\\text{end}}} \vec{E} \cdot d\vec{A}_{\text{end}} + \int_{\substack{\text{right}\\\text{end}}} \vec{E} \cdot d\vec{A}_{\text{end}} = \frac{q_{in}}{\varepsilon_0}$$

$$\Rightarrow \Phi_E = \int_{\text{side}} E dA \cos\theta_{\text{side}} + \int_{\substack{\text{left}\\\text{end}}} E dA \cos\theta_{\text{end}} + \int_{\substack{\text{right}\\\text{end}}} E dA \cos\theta_{\text{end}} = \frac{q_{in}}{\varepsilon_0}$$

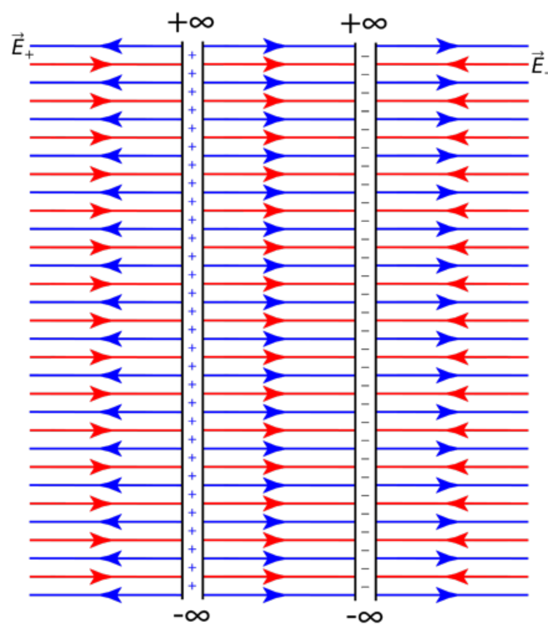
$$\Rightarrow \Phi_E = \int_{\text{side}} E dA \cos(90°) + \int_{\substack{\text{left}\\\text{end}}} E dA \cos(0°) + \int_{\substack{\text{right}\\\text{end}}} E dA \cos(0°) = \frac{q_{in}}{\varepsilon_0}$$

$$\Rightarrow \Phi_E = E \int_{\substack{\text{left} \\ \text{end}}} dA + E \int_{\substack{\text{right} \\ \text{end}}} dA = E\left(2A_{end}\right) = \frac{q_{in}}{\varepsilon_0}$$

$$\& \ \sigma = \frac{Q}{A} = \frac{q_{in}}{A_{end}} \Rightarrow q_{in} = \sigma A_{end} \Rightarrow E\left(2A_{end}\right) = \frac{\sigma A_{end}}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

Notice this electric field is uniform and is independent of the distance from the infinite plane of charges.

And notice what happens if we have two infinite parallel planes of charges, one with positive charge and one with negative charge:




The electric field outside the planes of charges cancels out to give zero electric field outside the planes of charges:

$$E_{outside} = 0$$

And between the two planes of charges, the electric fields add together:

$$E_{between} = 2E_{one\ plate} = 2\left(\frac{\sigma}{2\varepsilon_0}\right) = \frac{\sigma}{\varepsilon_0}$$

And we have begun our journey towards determining the capacitance of a parallel plate capacitor…

Notice that a positively charged particle moving through a constant electric field will experience an electrostatic force in the direction of the electric field. This force will be constant and equal to qE. In other words, the motion of a charged particle through a constant electric field will have similar characteristics to a mass moving through the constant gravitational field near the surface of a planet. This is very similar to projectile motion.

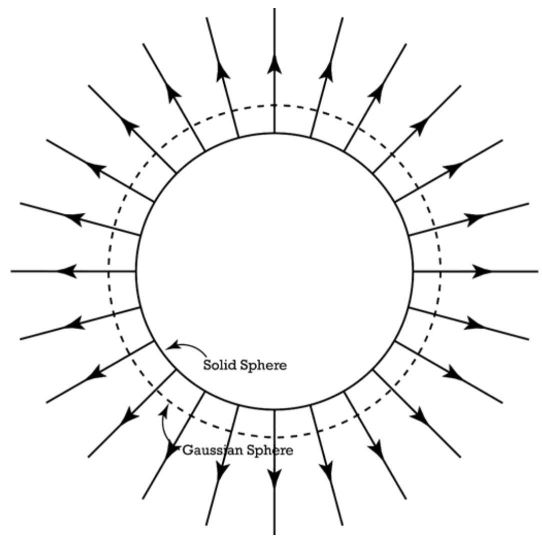
A few loose ends:

- According to the AP Physics C: Electricity and Magnetism Guidelines, you are responsible for a quantitative approach of Gauss' law to solve for electric fields only for charge distributions which are spherically, cylindrically, or planarly symmetric.
    a. In other words, to keep the math from getting overly complicated, you are responsible for solving equations with Gauss' law only in situations which are highly symmetrical.
    b. Some examples are:
        i. Inside and outside of a solid sphere made of conducting or insulating material. The Gaussian surface is a sphere.
        ii. Inside and outside of a thin spherical shell. Again, the Gaussian surface is a sphere.
        iii. An infinitely long, thin line of charges. The Gaussian surface is a cylinder that is colinear with the line of charges.
        iv. An infinitely large, thin plane of charges. We just did this. The Gaussian surface is a cylinder whose ends are parallel to the plane of charges.
        v. 2 infinitely large, thin parallel planes of charges. We just did this. Do a single plane of charges first.

- Do not forget the three kinds of charge densities which you are responsible for being able to use with Gauss' law depending on whether the charge is one, two, or three dimensional.
    a. linear charge density, $\lambda = \frac{Q}{L}$ in $\frac{C}{m}$
    b. surface charge density, $\sigma = \frac{Q}{A}$ in $\frac{C}{m^2}$
    c. volumetric charge density, $\rho = \frac{Q}{V}$ in $\frac{C}{m^3}$

Lastly, realize Gauss' law uses electric flux which is a measure of the number of imaginary electric field lines which pass through an imaginary Gaussian surface and those imaginary field lines are caused by highly symmetric groups of stationary point charges which are imperceptible to the naked eye. Yeah, it takes a bit of imagination to be able to visualize all of this. Which is why you need to practice!

A loose end:
- Outside the surface of a uniformly charged sphere, the electric field is the same as if the charged sphere were a point particle.
    - Example: Solid, uniformly charged sphere with charge Q and radius, a.
    - Create a Gaussian surface which is a concentric sphere with radius r > a.




$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

$$\Rightarrow \Phi_E = \int_{\text{sphere}} E \cos\theta \, dA = \int_{\text{sphere}} E \cos(0°) \, dA = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

$$\Rightarrow E \int_{\text{sphere}} dA = EA_{\text{sphere}} = E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow E = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q}{r^2} \Rightarrow E = \frac{kQ}{r^2}$$

    - This is true of a conductor or an insulator, however, the electric field inside a conductor will be zero, and inside an insulator the electric field depends on the radius and charge distribution, and can be derived in a similar manner.

Flipping Physics Lecture Notes:
Electric Potential
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-electric-potential.html

The electrostatic force is a conservative force, therefore:

$$F_x = -\frac{dU}{dx} \Rightarrow F_e = -\frac{dU_e}{dr} \Rightarrow dU_e = -\vec{F}_e \cdot d\vec{r} \Rightarrow \int dU_e = -\int \vec{F}_e \cdot d\vec{r}$$

$$\Rightarrow \Delta U_e = -\int \vec{F}_e \cdot d\vec{r} \ \ \& \ \ \vec{E} = \frac{\vec{F}_e}{q} \Rightarrow \vec{F}_e = q\vec{E} \Rightarrow \Delta U_e = -\int_A^B q\vec{E} \cdot d\vec{r}$$

$$\Rightarrow \Delta U_e = -q\int_A^B \vec{E} \cdot d\vec{r} = -q\int_A^B E\cos\theta\, dr$$

We have determined the change in electric potential energy experienced by a charged particle which has moved from point A to point B in an electric field. Notice, because the electrostatic force is a conservative force, this change in electric potential energy does not depend on the path taken from point A to point B.

To make a comparison to gravitational potential energy, if we lift an object vertically upward in a constant downward gravitational field, it will experience a positive change in gravitational potential energy.

The same is true for a positively charged object, if we move a positively charged object vertically upward in a constant downward electric field, it will experience a positive change in electric potential energy.

Notice the negative and the dot product in the equation. As we move a charge in a direction opposite the direction of the field, the direction of the displacement of the charge and the direction of the field are opposite to one another, therefore, the angle between those two directions is 180°, the cosine of 180° is negative one, which makes the change in potential energy of a positive charge positive.

$$\Rightarrow \Delta U_e = -q\int_A^B E\cos(180°)\, dr = -q\int_A^B E(-1)\, dr = q\int_A^B E\, dr$$

Next, we need to define **electric potential**. Just like we define the electric field in terms of the force experienced by a small, positive test charge, we define the electric potential in terms of the energy experienced by a small, positive test charge.

- $\vec{E} = \frac{\vec{F}_e}{q} \Rightarrow V = \frac{U_e}{q}$ in volts, $V = \frac{J}{C}$
- The symbol for electric potential is V. Yes, I know. The symbol for electric potential, V, is the same as the symbol for the units for electric potential, volts, V. It's not my fault.
- Electric potential is a **scalar** attribute of a **vector** electric field which does not depend on any electric charges which could be placed in that field.
    - The fact that electric potential is a scalar can be very helpful in this class.
    - This scalar can be either positive or negative for any given location.
    - Just like gravitational potential energy, we need to either assign a location where it equals zero, or follow a convention for assigning the zero potential location.
- Most often we work with **electric potential difference** not just electric potential. Electric potential difference is the difference in the electric potential between two points:[1]

[1] I know. I know. … Duh! … But it had to be said.

$$\Delta V = V_f - V_i = V_B - V_A \;\;\&\;\; V = \frac{U_e}{q}$$

$$\Rightarrow \Delta V = \frac{\Delta U_e}{q} = \frac{-q\int_A^B \vec{E}\cdot d\vec{r}}{q} = -\int_A^B \vec{E}\cdot d\vec{r}$$

- We will often set the initial electric potential, or electric potential at point A, equal to zero.
- Realize we can rearrange every integral to form a derivative:

$$\Delta V = -\int_A^B \vec{E}\cdot d\vec{r} \Rightarrow dV = -\vec{E}\cdot d\vec{r} \Rightarrow E_r = -\frac{dV}{dr}$$

Now that we have volts, the units for electric potential, it is important to realize the units for the electric field can be given in terms of volts as well.

$$\vec{E} = \frac{\vec{F}_e}{q} \text{ in } \frac{N}{C} = \left(\frac{N}{C}\right)\left(\frac{m}{m}\right) = \left(\frac{J}{C}\right)\left(\frac{1}{m}\right) = \frac{V}{m} \Rightarrow \frac{N}{C} = \frac{V}{m}$$

If a charge is moved from point A to point B via an external force, the external force does work on the charge, that changes the electric potential energy of the charge. And, as long as there is no change in the kinetic energy of the charge, that work equals the charge of the charge multiplied by the electric potential difference the charge went through: $W = q\Delta V$

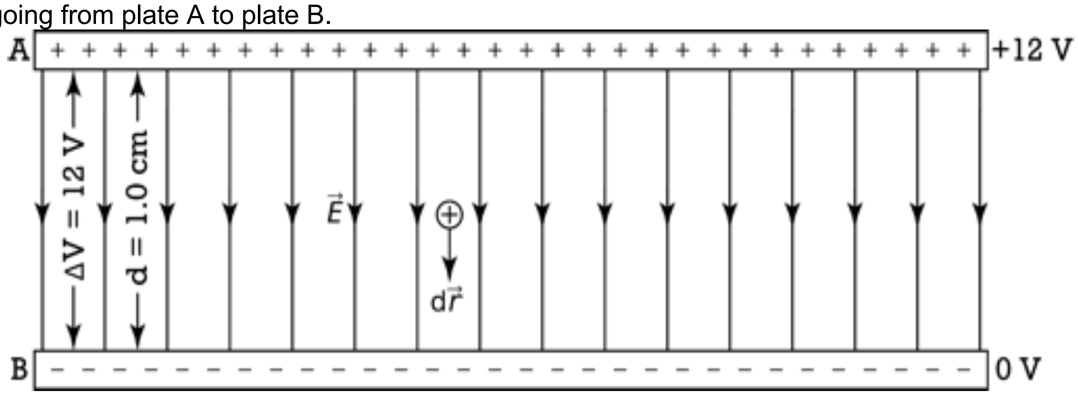
A unit of energy often used for very small amounts of energies, like one would use in atomic and nuclear physics, is the electron volt (eV). An electron volt is defined as the energy a charge-field system gains or losses when a charge of magnitude e (the elementary charge or the magnitude of the charge on an electron or proton) is moved through a potential difference of 1 V:

$$W = q\Delta V \Rightarrow W_{\text{eV}} = \left(1.6\times10^{-19}C\right)(1V) = 1.6\times10^{-19}C\cdot V \;\;\&\;\; C\cdot V = C\cdot\frac{J}{C} = J$$

$$\Rightarrow 1eV = 1.6\times10^{-19}J$$

I consider the electron volt to be a misnomer because it sounds like a unit of electric potential (volts), however, it is a unit of energy. It also refers to a positive amount of energy, even though the electron is negative. Be careful of that.

Let's say we have two, large, equal magnitude charged parallel plates, the top plate has a positive charge, and the bottom plate has a negative charge. We have shown the electric field is constant in this case and will be directed downward. Let's say the electric potential difference between the two plates is 12 volts and the distance between the two plates is 1.0 cm. Let's define the top plate as plate A, and the bottom plate as plate B. Let's start by determining the general equation for the electric potential difference when going from plate A to plate B.



$$\Delta V = -\int_A^B \vec{E} \cdot d\vec{r} = -\int_A^B E\cos\theta \, dr$$

$$\Rightarrow \Delta V = -\int_A^B E\cos(0°)\,dr = -E\int_A^B dr \Rightarrow \Delta V_{\text{constant E}} = -Ed$$

This is a good time to discuss the negative sign in the electric potential equation. In other words, a charge moving in the direction of the electric field will go through a negative potential difference and a charge moving opposite the direction of the electric field will go through a positive electric potential difference.

Add to the example: If we release a proton from the inside surface of plate A, what will the speed of the proton be right before it runs into plate B?

Set initial point at A and final point at B. Do not need a horizontal zero line because gravitational potential energy for subatomic particles is usually negligible and it is in this case. We do not actually know the electric potential energy initial or electric potential energy final; however, we do know the change in the electric potential energy. Also, the charge and mass of a proton are given in the Table of Information provided on the AP Physics C exam.

$$ME_i = ME_f \Rightarrow ME_A = ME_B \Rightarrow U_{\text{elec}_A} = KE_B + U_{\text{elec}_B}$$

$$\Rightarrow -KE_B = U_{\text{elec}_B} - U_{\text{elec}_A} = \Delta U_{\text{elec}} \Rightarrow -\frac{1}{2}mv_B^2 = q\Delta V$$

$$\& \ \ \Delta V = \frac{\Delta U_{\text{elec}}}{q} \Rightarrow \Delta U_{\text{elec}} = q\Delta V \ \ \& \ \ \Delta V_{A\to B} = -12V$$

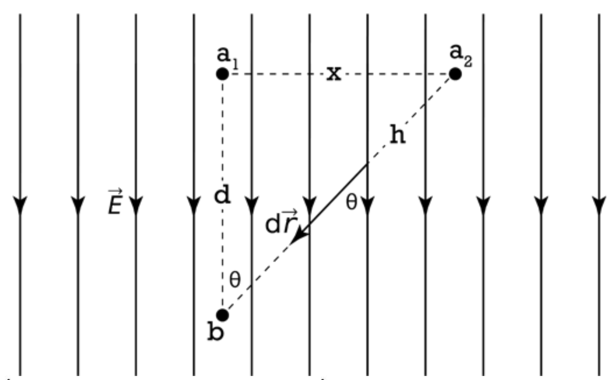
$$\Rightarrow v_B = \sqrt{-\frac{2q\Delta V}{m}} = \sqrt{-\frac{(2)\left(1.6\times10^{-19}\right)(-12)}{1.67\times10^{-27}}} = 47952\frac{m}{s} \approx 48\frac{km}{s}$$

$$v_B = 47952\frac{m}{s}\left(\frac{3600\ s}{1\ \text{hr}}\right)\left(\frac{1\ \text{mi}}{1609\ m}\right) = 107288\frac{\text{mi}}{\text{hr}} \approx 1.1\times10^5\frac{\text{mi}}{\text{hr}}$$

Now let's look at determining the electric potential difference when moving at an angle relative to a uniform electric field. We already know the electric potential difference when moving from point $a_1$ to b:

$$\Delta V_{a_1\to b} = -Ed$$

Let's determine the electric potential difference when moving from point $a_2$ to b:




$$\Delta V_{a_2\to b} = -\int_{a_2}^b E\cdot dr = -\int_{a_2}^b E\cos\theta\,dr = -\int_{a_2}^b E\left(\frac{d}{h}\right)dr = -\left(\frac{Ed}{h}\right)\int_{a_2}^b dr$$

$$\Rightarrow \Delta V_{a_2 \to b} = -E\left(\frac{d}{h}\right)(h) = -Ed \Rightarrow \Delta V_{a_1 \to b} = \Delta V_{a_2 \to b}$$

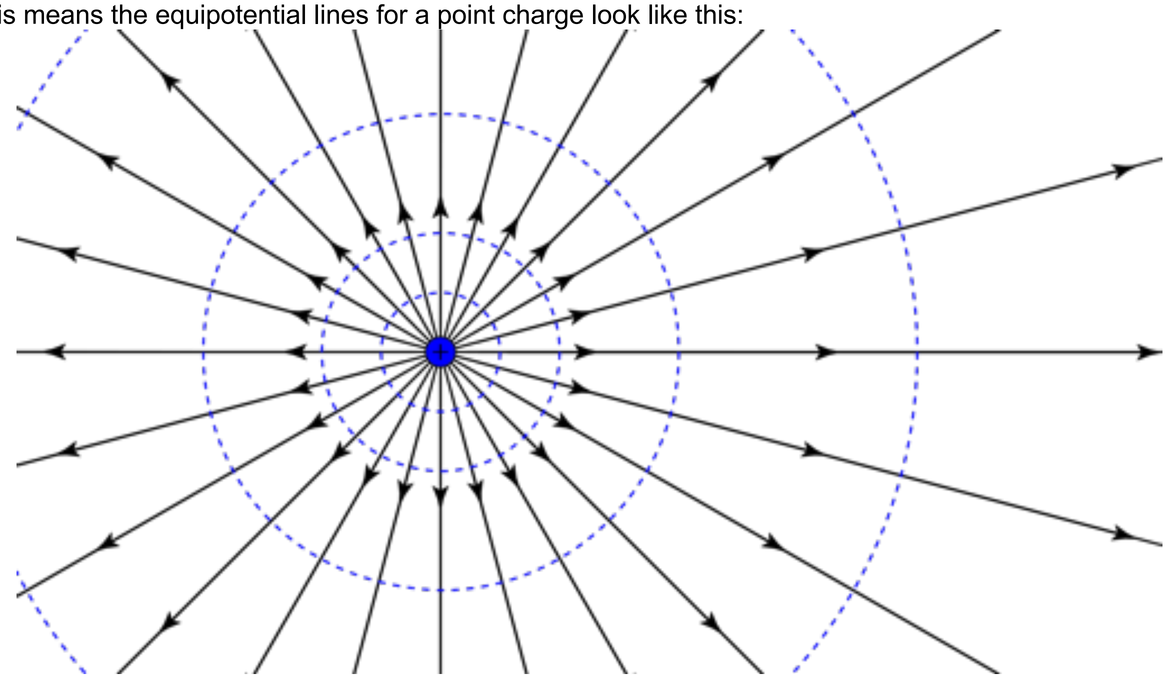
The electric potential difference is the same for both of these because points $a_1$ and $a_2$ have the same electric potential.

$$\Delta V_{a_2 \to a_1} = -\int_{a_2}^{a_1} E \cdot \mathrm{d}r = -E\int_{a_2}^{a_1} \cos\theta \, \mathrm{d}r = -E\int_{a_2}^{b} \cos(90°)\, dr = 0$$

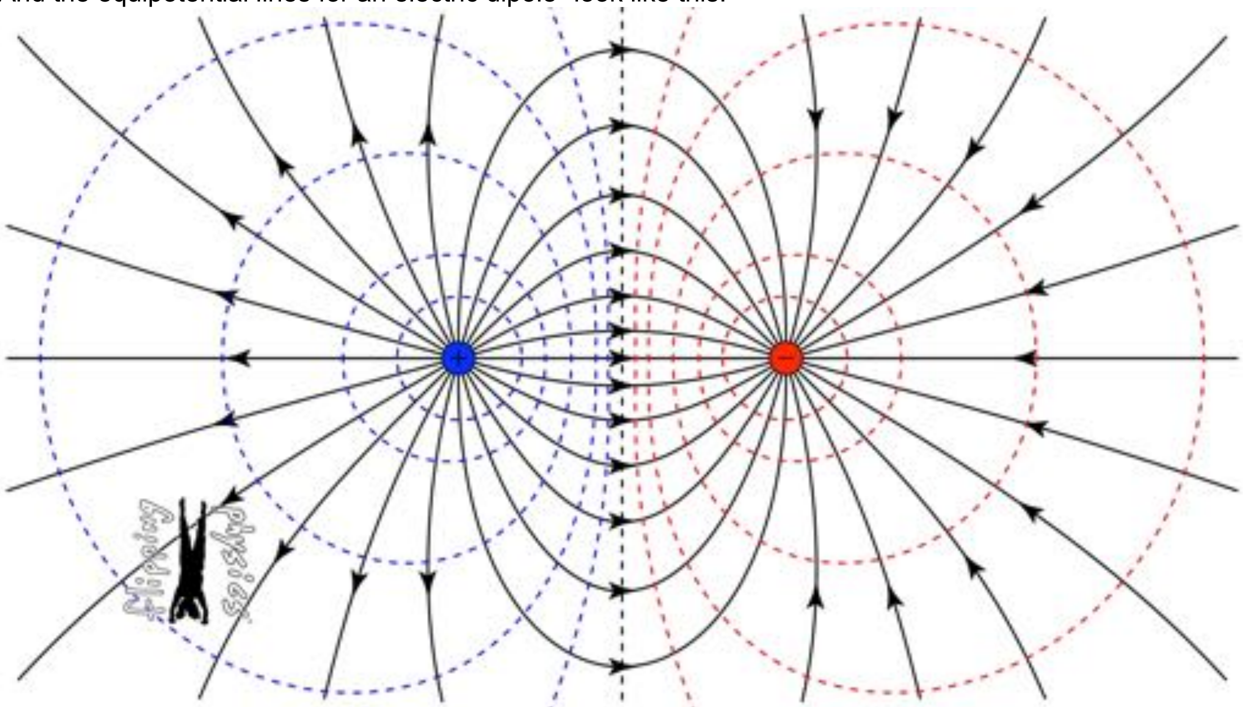
Points $a_1$ and $a_2$ are on an equipotential surface. An equipotential surface (or line):

- Has the same electric potential at every point on the surface (or line)
- Is always perpendicular to the electric field
  - Therefore, the electric field has no component along the equipotential line
- Equipotential lines are sometimes called isolines
- And it takes zero work to move a charged object along an equipotential surface
  - $W = q\Delta V \Rightarrow W_{\text{equipotential surface}} = q(0) = 0$

This means the equipotential lines for a point charge look like this:



And the equipotential lines for an electric dipole[2] look like this:



The equation for the electric potential which surrounds and is caused by a point charge is:

$$V_{\text{point charge}} = \frac{kq}{r}$$

This equation assigns our location of zero electric potential to be infinitely far away.

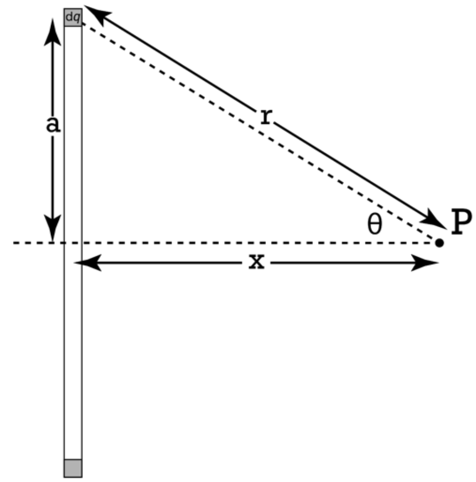
We can use the relationship between electric potential and electric potential energy to determine the electric potential energy which surrounds and is caused by a point charge:

$$V = \frac{U_{\text{elec}}}{q} \Rightarrow U_{\text{elec}} = qV \Rightarrow U_{\text{2 point charges}} = q_1\left(\frac{kq_2}{r}\right)$$

$$\Rightarrow U_{\text{2 point charges}} = \frac{kq_1q_2}{r}$$

[2] This is a simple example of an electric dipole which is a pair of electric charges of equal magnitude, but opposite sign separated by some typically small distance.

Realize, because electric potential and electric potential energy are scalar values, determining those values for multiple particles uses superposition. You just add all the values together.




In order to understand how useful it is that electric potential is a scalar and not a vector, let's revisit an example from before. Let's determine the electric potential caused by a uniformly charged, thin ring of charge +Q, with radius a, at point P, which is located on the axis of the ring a distance x from the center of the ring.

Because this is a continuous charge distribution, we need to break the uniformly charged thin ring of charge +Q into an infinite number of infinitesimally small charges, dQ.

$$V_{\text{point charge}} = \frac{kq}{r} \Rightarrow V_{\text{continuous charge distribution}} = \int \left(\frac{k}{r}\right) \mathrm{d}q = \frac{k}{r}\int \mathrm{d}q = \frac{kQ}{r} \Rightarrow V_P = \frac{kQ}{\sqrt{a^2 + x^2}}$$

And from there we can determine the electric field at point P.

$$E_r = -\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{kQ}{\sqrt{a^2+x^2}}\right) = -kQ\frac{\mathrm{d}}{\mathrm{d}x}\left(a^2+x^2\right)^{-\frac{1}{2}} = -kQ\left(-\frac{1}{2}\right)\left(a^2+x^2\right)^{-\frac{3}{2}}(2x)$$

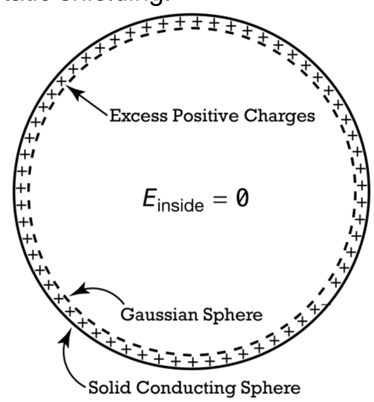
$$\Rightarrow E_P = \frac{kQx}{\left(a^2+x^2\right)^{\frac{3}{2}}}$$

Notice that this derivation of the electric field at point P is much easier than deriving the electric field directly like we did before. Therefore, I would recommend that you remember that, for a continuous charge distribution, you can first determine the electric potential and then the electric field, and that is often easier than solving for the electric field directly.

Flipping Physics Lecture Notes:
Conductors in Electrostatic Equilibrium
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-conductors-electrostatic-equilibrium.html

Conductors are materials where the electrons are free to move rather easily, however, when they are in electrostatic[1] equilibrium, this means the charges are stationary in the object. There are four things you need to remember about conductors in electrostatic equilibrium:

1) The electric field inside a conductor in electrostatic equilibrium equals zero. $E_{inside} = 0$
   a. If the electric field inside were not equal to zero, charges would have a net electrostatic force acting on them and they would accelerate, therefore the conductor would not be in electrostatic equilibrium.
      i. $E_{inside} \neq 0 \Rightarrow F_e = qE \neq 0 \Rightarrow$ not in electrostatic equilibrium
   b. Notice that this means that anything inside a conductor in electrostatic equilibrium is shielded from all external electric fields. This is called electrostatic shielding.
2) All excess charges are located on the surface (or surfaces) of the conductor.
   a. Solid conducting sphere example:
      i. Draw a Gaussian surface as a concentric sphere with a radius slightly smaller than the radius of the sphere.
      ii. Using Gauss' law, because there is no electric field inside the conductor in electrostatic equilibrium, we know the left-hand side of the equation equals zero.
      iii. Therefore, there must be zero net charge inside the Gaussian sphere and all the excess charges must be outside the Gaussian sphere.
      iv. Therefore, all the excess charges are on the surface of the conductor.




$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0} \ \& \ E_{inside} = 0 \Rightarrow 0 = \frac{q_{enclosed}}{\varepsilon_0} \Rightarrow q_{enclosed} = 0$$

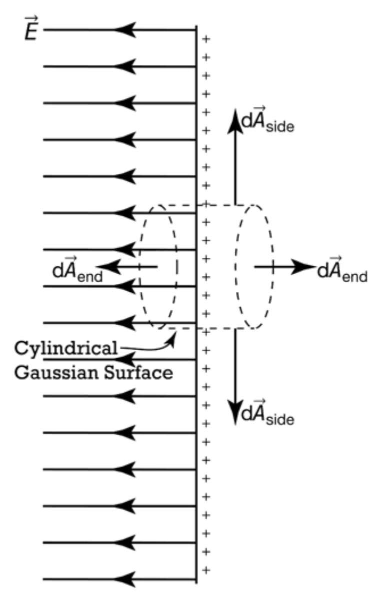
3) The electric field just outside the surface of a conductor in electrostatic equilibrium is:

$$E_{\substack{just \\ outside}} = \frac{\sigma_{local}}{\varepsilon_0} \ \& \ \perp \text{ to surface}$$

   a. If the electric field had a component parallel to the surface of the conductor, the charges would move, and the conductor would no longer be in electrostatic equilibrium. Therefore, the electric field at the surface of a conductor in electrostatic equilibrium must be perpendicular to the surface.
      i. Because equipotential surfaces are always perpendicular to the electric field, the surface of a conductor in electrostatic equilibrium must be an equipotential surface.

$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{r} = -\int_A^B E\cos\theta dr = -\int_A^B E\cos(90°) dr = 0$$

   b. If we zoom way in on the surface of the conductor in electrostatic equilibrium, we can draw a Gaussian cylinder




[1] Electrostatics is the study of electromagnetic phenomena that occur when there are no moving charges.

with its cylindrical axis normal to the surface of the conductor.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

$$\Rightarrow \Phi_E = \int_{\text{side}} E dA \cos\theta_{\text{side}} + \int_{\substack{\text{left}\\\text{end}}} E dA \cos\theta_{\text{end}} + \int_{\substack{\text{right}\\\text{end}}} E dA \cos\theta_{\text{end}} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

$$\Rightarrow \int_{\text{side}} E dA \cos(90°) + \int_{\substack{\text{left}\\\text{end}}} E dA \cos(0°) + \int_{\substack{\text{right}\\\text{end}}} (0)\, dA \cos\theta_{\text{end}} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

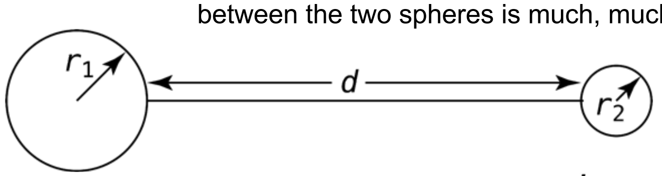
$$\& \ \sigma = \frac{Q}{A} \Rightarrow \sigma_{\text{local}} = \frac{q_{\text{in}}}{A_{\text{end}}} \Rightarrow q_{\text{in}} = \sigma_{\text{local}} A_{\text{end}}$$

$$\Rightarrow E \int_{\substack{\text{left}\\\text{end}}} dA = EA_{\text{end}} = \frac{\sigma_{\text{local}} A_{\text{end}}}{\varepsilon_0} \Rightarrow E = \frac{\sigma_{\text{local}}}{\varepsilon_0}$$

4) For an irregular shape, the local surface charge density is at its maximum where the radius of curvature is at its minimum. In other words, the largest number of excess charges per area will be where the radius of curvature is the smallest.

$$\sigma_{\text{local}} = \text{maximum @ } r_{\text{curvature}} = \text{minimum}$$

   a. To prove this, we have two conducting spheres connected by a long conducting wire with the whole system in electrostatic equilibrium.
      i. This system is a conductor in electrostatic equilibrium. In other words, when two conductors are brought into contact with one another, the charges redistribute such that both conductors are at the same electric potential. Please realize this happens so quickly that the time for this to occur is considered to be negligible.
   b. The radius of sphere 2 is smaller than the radius of sphere 1, and the distance, d, between the two spheres is much, much larger than either radius.




$$r_2 < r_1 \ \& \ d >> r_1 \ \& \ V_1 = V_2 \Rightarrow \frac{kq_1}{r_1} = \frac{kq_2}{r_2} \Rightarrow \frac{q_1}{r_1} = \frac{q_2}{r_2}$$

$$\Rightarrow q_1 = \left(\frac{r_1}{r_2}\right) q_2 \ \& \ \frac{r_1}{r_2} > 1 \Rightarrow q_1 > q_2$$

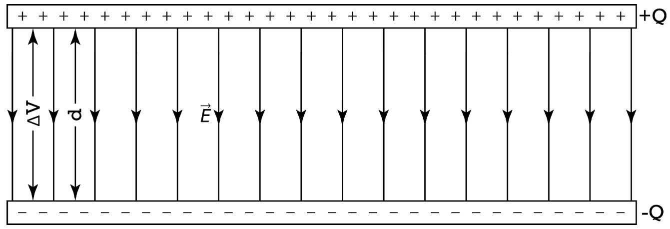
$$E_1 = \frac{kq_1}{(r_1)^2} \ \& \ E_2 = \frac{kq_2}{(r_2)^2} \Rightarrow \frac{E_1}{E_2} = \frac{\frac{kq_1}{(r_1)^2}}{\frac{kq_2}{(r_2)^2}} = \left(\frac{kq_1}{(r_1)^2}\right)\left(\frac{(r_2)^2}{kq_2}\right) = \frac{q_1 (r_2)^2}{q_2 (r_1)^2}$$

$$\Rightarrow \frac{E_1}{E_2} = \frac{\left(\left(\frac{r_1}{r_2}\right) q_2\right)(r_2)^2}{q_2 (r_1)^2} = \frac{r_2}{r_1} \Rightarrow E_2 = \left(\frac{r_1}{r_2}\right) E_1 \ \& \ \frac{r_1}{r_2} > 1$$

$$\Rightarrow E_2 > E_1 \ \& \ E = \frac{\sigma_{\text{local}}}{\varepsilon_0} \Rightarrow \sigma_2 > \sigma_1 \Rightarrow \text{if } r_2 < r_1 \text{ then } \sigma_2 > \sigma_1$$

Flipping Physics Lecture Notes:
Capacitors
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-capacitors.html

A capacitor is a way to store electric potential energy in an electric field. The simplest form of a capacitor is a parallel plate capacitor.




Capacitance, C, is defined as the magnitude of the charge stored on one plate divided by the electric potential difference between the two plates: $C \equiv \frac{Q}{\Delta V}$

- Capacitance is always positive.
  - Q, is the charge on the positive plate.
  - ΔV is the positive electric potential difference between the two plates.
- The net charge on a capacitor is zero.
  - $Q_{\text{total}} = +Q + (-Q) = 0$
- $C \equiv \frac{Q}{\Delta V} \Rightarrow \text{Capacitance in } \frac{\text{coulombs}}{\text{volts}} = F\text{, farads}$
  - $\text{charge, } Q \Rightarrow \text{coulombs, } C \;\;\&\;\; \text{capacitance, } C \Rightarrow \text{farads, } F$
  - It is not my fault the symbol for capacitance is C and capacitance is charge per electric potential difference and the units for charge are coulombs for which the symbol is C.
- The three-line equal sign, $\equiv$, means "is defined as". This is not a derivation. We made it up. We have simply decided to define the charge on a capacitor divided by the electric potential difference of the capacitor as "capacitance".
- Energy is stored in the electric field of the capacitor.
- The capacitance of a capacitor depends only on the capacitor's physical characteristics. For example, the capacitor's shape and material used to separate the plates of the capacitor.

The basic idea is:

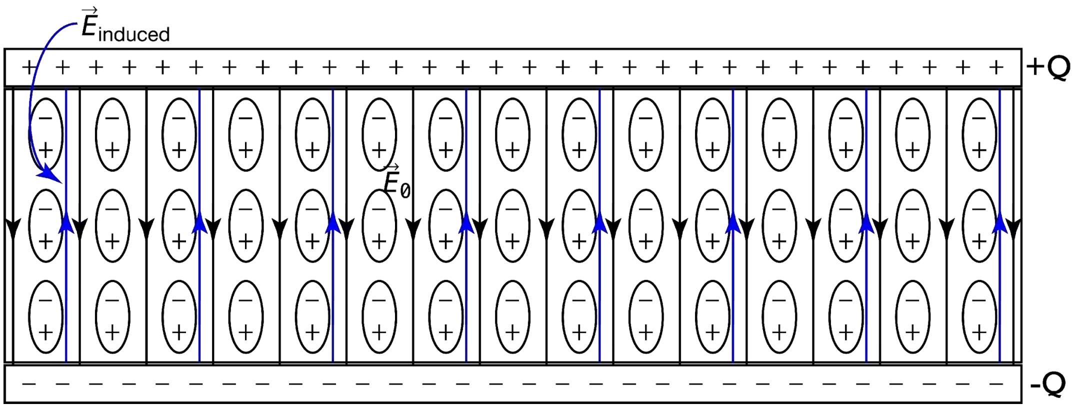
- Start with an uncharged capacitor.
  - No charge on either plate.
  - No electric field between the plates.
- Attach the terminals of a battery to the two plates of the capacitor.
- Charges flow from one plate to the other plate of the capacitor.
- We now have a charged capacitor.
  - Both plates have equal magnitude charge.
  - There is an electric field and an electric potential difference between the plates.
  - Energy is stored in the electric field of the capacitor.

Let's derive the equation for the capacitance of a parallel plate capacitor. We have already derived two equations for two parallel, infinitely large, charged plates with equal magnitude, but opposite sign.

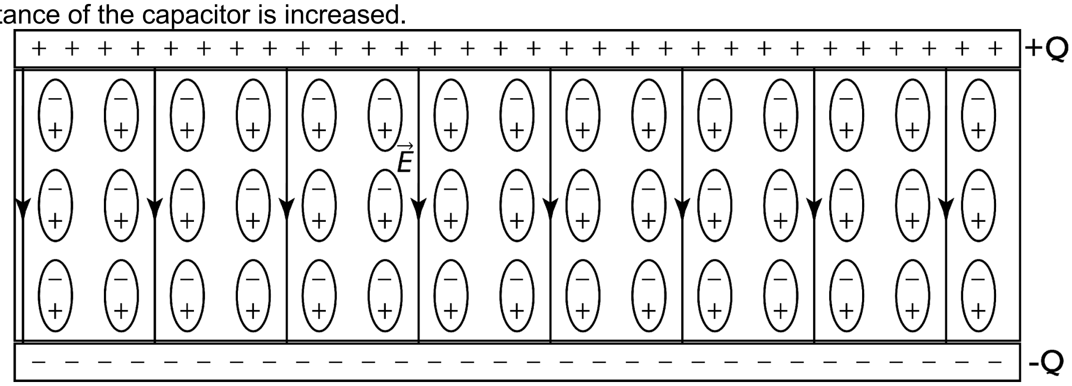
$$E_{\parallel \text{ plates}} = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0} \;\;\&\;\; \Delta V_{\text{constant E}} = -Ed \Rightarrow \|\Delta V\| = Ed = \left(\frac{Q}{A\varepsilon_0}\right)d$$

$$\& \;\; C = \frac{Q}{\Delta V} = \frac{Q}{\frac{Qd}{A\varepsilon_0}} \Rightarrow C_{\parallel \text{ plate}} = \frac{\varepsilon_0 A}{d}$$

This assumes there is a vacuum between the two plates. Usually, we place an insulating material between the plates of a capacitor. This is both to help physically separate the two plates and because it increases the capacitance of the capacitor. This insulating material is called a dielectric.




The charged particles in the dielectric are polarized and induce their own electric field (above in blue) which is opposite the direction of the original electric field of the capacitor $E_0$ (above in black). The net electric field (below in black) is decreased. Because the electric field is decreased, the electric potential difference across the capacitor is decreased, the charge of the capacitor remains the same, and the capacitance of the capacitor is increased.




$$E = E_0 - E_{\text{induced}} \Rightarrow E \downarrow \;\&\; \|\Delta V\| = Ed$$
$$\Rightarrow \Delta V \downarrow \;\&\; Q \text{ is constant} \;\&\; C = \frac{Q}{\Delta V} \Rightarrow C \uparrow$$

The way we define the effect of a dielectric is with the dielectric constant. The symbol for the dielectric constant is the lowercase Greek letter kappa, κ. It looks basically like a lowercase k. The dielectric constant equals the ratio of the electric permittivity of the dielectric to the electric permittivity of free space.

- $$\kappa = \frac{\varepsilon}{\varepsilon_0}$$
- The dielectric constant has no units.
- Electric permittivity is the measurement of how much a material is polarized when it is placed in an electric field.
  - The easier it is for electrons to change configurations in a material, the larger the dielectric constant of that material.
- The dielectric constant is also sometimes called *relative permittivity*.

We can also determine the relationship between the electric field between the parallel plates of the capacitor with a vacuum and with a dielectric.

$$E_{\text{vacuum}} = \frac{\sigma}{\varepsilon_0} \;\&\; E_{\text{dielectric}} = \frac{\sigma}{\varepsilon} \Rightarrow \frac{E_{\text{vacuum}}}{E_{\text{dielectric}}} = \frac{\frac{\sigma}{\varepsilon_0}}{\frac{\sigma}{\varepsilon}} = \frac{\varepsilon}{\varepsilon_0} = \kappa \Rightarrow \kappa = \frac{E_{\text{vacuum}}}{E_{\text{dielectric}}} \Rightarrow \kappa = \frac{E_0}{E}$$

And then use that to determine the relationship between the capacitance of the capacitor with a vacuum and the capacitance of the capacitor with a dielectric.

$$C_{\parallel\text{ plate}} = C_0 = \frac{\varepsilon_0 A}{d} \ \ \& \ \ \kappa = \frac{\varepsilon}{\varepsilon_0} \Rightarrow \varepsilon = \kappa\varepsilon_0$$

$$\& \ \ C_{\text{dielectric}} = C = \frac{\varepsilon A}{d} = \frac{\kappa\varepsilon_0 A}{d} \Rightarrow C_{\text{dielectric}} = \frac{\kappa\varepsilon_0 A}{d} \ \ \& \ \ C = \kappa C_0$$

According to the College Board, students are responsible for determining the capacitance only of the following shapes: parallel-plate capacitors, spherical capacitors, and cylindrical capacitors.

Next, let's derive the equation for the energy stored in a capacitor. Starting with an uncharged capacitor, we move one, infinitesimally small charge from one plate to the other plate. Because the electric potential difference between the plates is zero, moving this first charge takes no work. However, moving the next charge does take work because there is now an electric potential difference between the two plates. The work it takes to move a charge equals the change in electric potential energy of the capacitor and it equals the magnitude of the charge which is moved times the electric potential difference the charge is moved through which is the electric potential difference across the capacitor which now has an infinitesimally small electric potential difference across it. We need to identify the infinitesimally small charge as dq and the amount of work it takes to move that charge dW. And take the integral of both sides.

$$Q_i = 0 \ \ \& \ \ W = \Delta U_{\text{elec}} = q\Delta V \Rightarrow \mathrm{d}W = \Delta V \mathrm{d}q$$

$$\& \ \ C = \frac{Q}{\Delta V} \Rightarrow \Delta V = \frac{Q}{C} \Rightarrow Q = C\Delta V$$

$$W = \int_0^Q \Delta V\,\mathrm{d}q = \int_0^Q \frac{q}{C}\mathrm{d}q = \left[\frac{q^2}{2C}\right]_0^Q = \frac{Q^2}{2C} - \frac{0^2}{2C} \Rightarrow U_C = \frac{Q^2}{2C}$$

$$\Rightarrow U_C = \frac{(C\Delta V)^2}{2C} = \frac{1}{2}C\Delta V^2 = \frac{1}{2}\left(\frac{Q}{\Delta V}\right)\Delta V^2 = \frac{1}{2}Q\Delta V$$

$$\Rightarrow U_C = \frac{Q^2}{2C} = \frac{1}{2}C\Delta V^2 = \frac{1}{2}Q\Delta V$$

Note: The energy stored in the capacitor is stored in the electric field of the capacitor and is equal to the amount of work needed to move the charges from one plate to the other.

The capacitor in the disposable camera:

$$C = 120\mu F;\ \Delta V = 330V$$

$$\Rightarrow U_C = \frac{1}{2}C\Delta V^2 = \frac{1}{2}\left(120 \times 10^{-6}\right)(330)^2 = 6.532 \approx 6.5J$$

$$\Rightarrow U_C = 6.532J \times \left(\frac{1\,eV}{1.6x10^{-19}J}\right) = 4.0825 \times 10^{19} \approx 41 \times 10^{18}eV$$

$$\Rightarrow U_C \approx 41 \times 10^9 \times 10^9 eV \Rightarrow U_C \approx 41 \text{ billion billion } eV$$

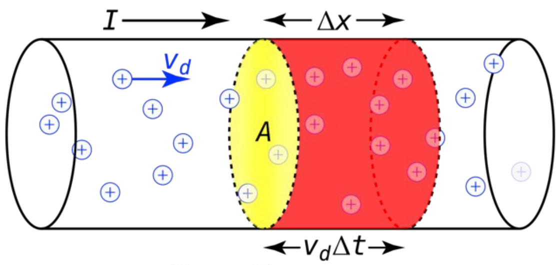
Flipping Physics Lecture Notes:
Current, Resistance, and Simple Circuits
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-current-resistance-simple-circuits.html

*Electric current*, I, is defined as the derivative of charge with respect to time: $I \equiv \frac{dq}{dt}$

- $I \equiv \frac{dq}{dt} \Rightarrow \frac{\text{coulombs, } C}{\text{seconds, } s} = \text{amperes, } A$
    - Amperes are a base S.I. unit.
    - This is instantaneous current.
- $I_{\text{average}} = \frac{\Delta Q}{\Delta t}$
- Current is the electric charge of the charges which pass by a point in a current carrying wire divided by the time it takes for those charges to pass by that point.
- Current occurs when there is an electric potential difference across a wire. If there is no electric potential difference, current does not flow.
    - $\Delta V = 0 \Rightarrow I = 0$

Unless otherwise stated, electric current in this class is all considered to be *conventional current*:

- The direction of conventional current is the direction positive charges *would* flow.
- Reality is that, in most circuits, negative charge carries (electrons, $e^-$) move opposite the direction of conventional current.




Let's look at charges flowing in a wire:
Start with the average current over a small section of the wire Δx:

- $I_{\text{average}} = \frac{\Delta Q}{\Delta t}$
- $\Delta Q = (\text{\# of charge carriers})(\text{charge per carrier, } q)$
- Charge carrier density, n:
- $n = \frac{\text{\# of charge carriers}}{\text{volume, } V} \Rightarrow \text{\# of charge carriers} = nV$
  $\Rightarrow \Delta Q = nVq \ \ \& \ \ V = A\Delta x \Rightarrow \Delta Q = nA\Delta x q$
- $v_{\text{drift}} = v_d = \frac{\Delta x}{\Delta t} \Rightarrow \Delta x = v_d \Delta t$
    - Drift velocity, $v_d$: The average velocity of the charge carriers in a current carrying wire.
        - If the current is zero, the charge carriers are still moving, however, the average velocity of the charge carriers is zero.
        - $v_d$ typically is quite low. On the order of 0.1 mm/s. The reason lightbulbs in a circuit (for example) turn on immediately when you flip the switch is because all the electrons are already in the wire. When you flip the switch, they all start flowing.
- $\Rightarrow \Delta Q = nAv_d\Delta t q \Rightarrow I = \frac{\Delta Q}{\Delta t} = \frac{nAv_d\Delta t q}{\Delta t} \Rightarrow I = nAv_d q$

*Current density*, J, is current per unit area:

- $J = \frac{I}{A} = \frac{nAv_d q}{A} \Rightarrow J = nv_d q$ & $J = \sigma E$
    - Materials which have this property are considered to be ohmic and follow Ohm's Law.
    - σ is the conductivity of the material.
        - Conductivity is a measure of how little a material opposes the movement of electric charges.
        - Conductivity is a fundamental property of a material.
- $\|\Delta V\| = Ed \Rightarrow \|\Delta V\| = EL$
    - An electric potential difference across a wire is what causes current in the wire and we are assuming the electric field created in the wire is uniform. Rather than using d for the distance in the electric field, we use L for the length of the wire.
- $\Rightarrow E = \frac{\Delta V}{L} \Rightarrow J = \sigma\left(\frac{\Delta V}{L}\right) \Rightarrow \Delta V = \frac{JL}{\sigma} = \frac{IL}{A\sigma} \Rightarrow \Delta V = \left(\frac{L}{\sigma A}\right)I$

The *resistance* of a wire, R, is defined as $R = \frac{L}{\sigma A}$

- However, usually resistance is defined in terms of *resistivity*, ρ.
    - Resistivity is a measure of how strongly a material opposes the movement of electric charges.
    - Resistivity is a fundamental property of a material.
    - $\rho = \frac{1}{\sigma} \Rightarrow R = \frac{\rho L}{A}$ & $E = \rho J$
    - This equation requires the resistor to have uniform geometry.
- Which brings us to the more common version of Ohm's law:
    - $\Delta V = \left(\frac{L}{\sigma A}\right)I = I\left(\frac{\rho L}{A}\right) \Rightarrow \Delta V = IR$
    - Again, not all materials are ohmic and follow Ohm's law.
- $\Rightarrow R = \frac{\Delta V}{I} \Rightarrow \text{ohms}, \Omega = \frac{\text{volts}, V}{\text{amperes}, A}$
- $R = \frac{\rho L}{A} \Rightarrow \rho = \frac{RA}{L} \Rightarrow \frac{\Omega \cdot m^2}{m} = \Omega \cdot m$

*Resistance* and *resistivity* are two terms which students often mix up:

- Resistance has units of ohms, Ω, and is a property of an object.
- Resistivity has units of Ω·m and is property of a material.
- Two objects can have the same resistivity but different resistances if they are made of the same material; however, they have different lengths or cross-sectional areas.

The resistivity of a conducting material typically decreases with decreasing temperature. Think of superconductors. Superconducting materials have zero resistivity, and require very, very low temperatures.

- In this class, unless otherwise stated, the resistivity of conducting materials is considered to be constant regardless of temperature.
- Resistors usually convert electric potential energy to thermal energy which can increase the temperature of the resistor and can increase the temperature of the resistor's environment.

Now we get to discuss *electric power*, which is the rate at which electric potential energy is converted to other types of energy such as heat, light, and sound.

$$P = \frac{dU}{dt} \Rightarrow P_{\text{elec}} = \frac{dU_{\text{elec}}}{dt} = \frac{d(q\Delta V)}{dt} = \frac{dq}{dt}\Delta V \Rightarrow P = I\Delta V$$

$$\& \;\; \Delta V = IR \Rightarrow P = I(IR) = I^2 R$$

$$\& \;\; I = \frac{\Delta V}{R} \Rightarrow P = \left(\frac{\Delta V}{R}\right)^2 R = \frac{\Delta V^2}{R}$$

$$\Rightarrow P = I\Delta V = I^2 R = \frac{\Delta V^2}{R}$$

A unit which is often used when it comes to electricity is the kilowatt-hour:

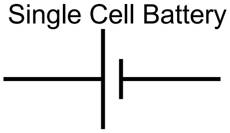
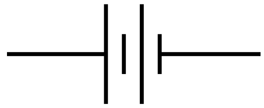
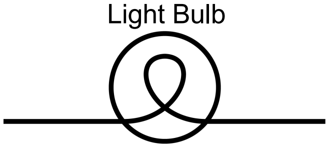
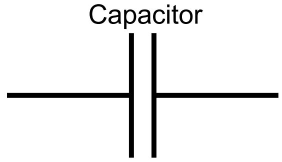
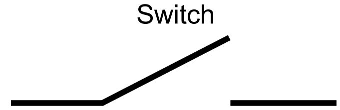
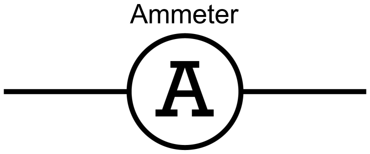
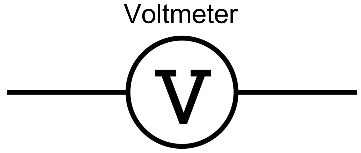
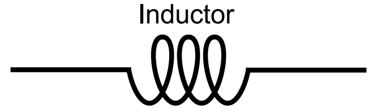
$$1kW{\cdot}hr\left(\frac{1W}{1000kW}\right) = 1000W{\cdot}hr = 1000\left(\frac{J}{s}\right)hr\left(\frac{3600s}{1hr}\right) = 3.6\times10^6 J$$

In other words, the kilowatt-hour is a misnomer (or maybe just misleading). It sounds like a unit of power; however, it is a unit of energy. And we know: $1kW \cdot hr = 3.6MJ$

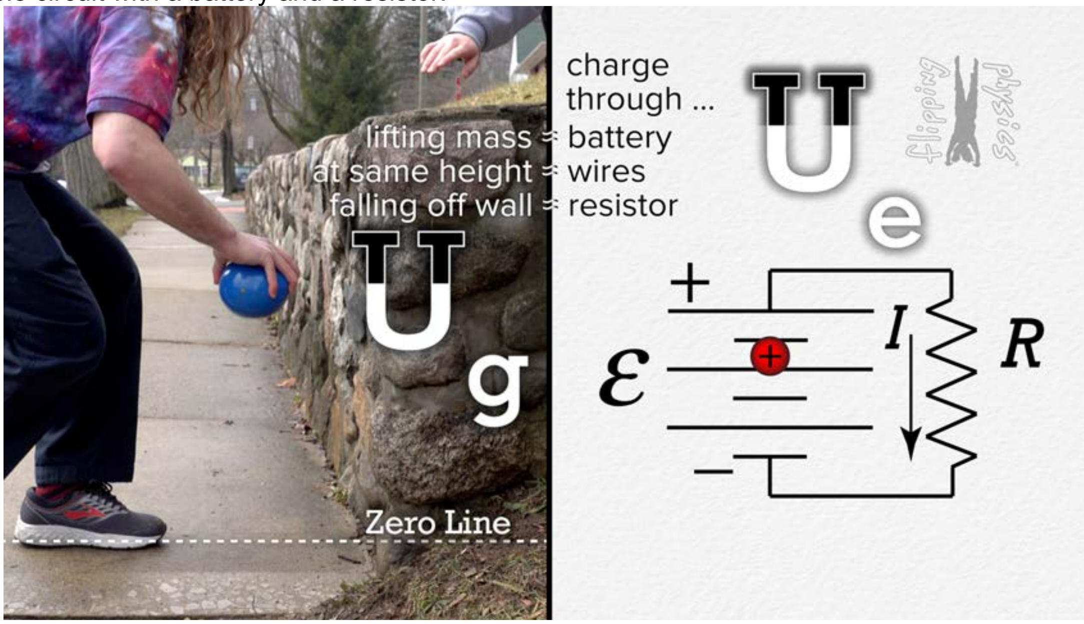
A light bulb is a common item used in physics. It is a resistor which converts electric potential energy to light, heat, and sound energy. The brightness of a light bulb increases with increasing power; therefore, the brightness of a light bulb is often used to demonstrate the power in an electric circuit. Speaking of electric circuits…

The Basics of Electric Circuits:

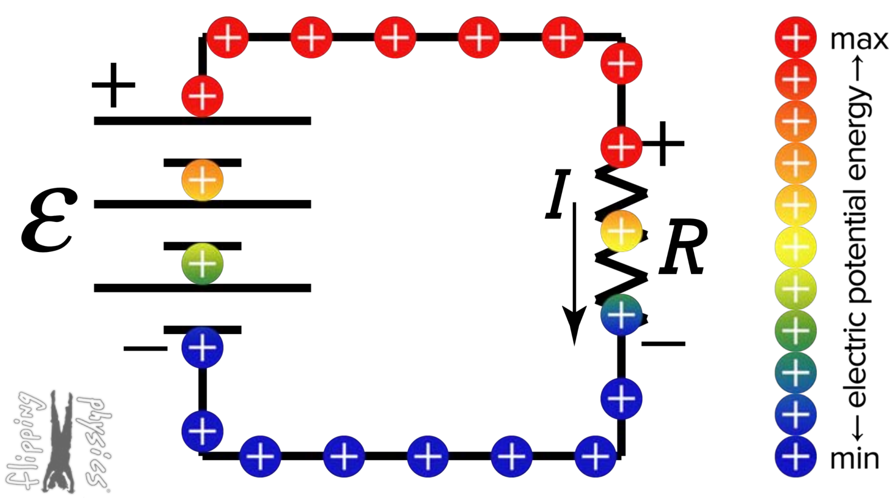
- An electric circuit is typically composed of electrical loops which can include wires, batteries, resistors, light bulbs, capacitors, switches, ammeters, voltmeters, and inductors.
- Typical symbols for elements in electric circuits are:

| | | |
|---|---|---|
| Single Cell Battery  | Double Cell Battery  | Resistor  |
| Light Bulb  | Capacitor  | Switch  |
| Ammeter   | Voltmeter   | Inductor  |

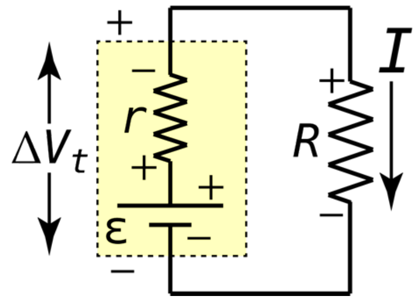
A simple circuit with a battery and a resistor:




- The long line of the battery is the positive terminal, and the short line is the negative terminal.
- *Electromotive force*, emf, ε, is the ideal electric potential difference, or voltage, across the terminals of the battery.
  - Yes, the symbol, lowercase Greek letter epsilon, is the same as electric permittivity. ☹
  - Yes, electromotive force is not a force. The term is another misnomer. ☹
- According to the law of charges, positive charges are repelled from the positive terminal and attracted to the negative terminal; therefore, current is clockwise in this circuit.
- A battery adds electric potential energy to electric charges.
  - Like lifting a mass adds gravitational potential energy to masses.
  - A battery is essentially an electric potential energy pump.
- A resistor converts electric potential energy to heat energy. (And maybe light sound energy)
  - Like a mass falling off a wall converts gravitational potential energy to kinetic energy.
- Unless otherwise stated, wires are considered to be ideal and have zero resistance; therefore, there is no change in electric potential energy of charges as they move along a wire.
  - Like a mass at rest maintaining a constant height and therefore a constant gravitational potential energy at either the top or bottom of the wall.

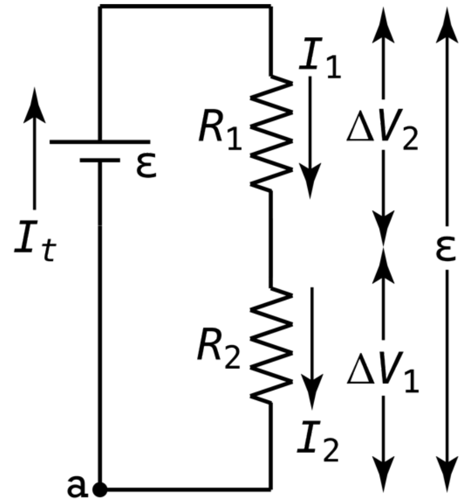


- *Terminal Voltage*, $\Delta V_t$, is the measured voltage across the terminals of the battery.
    - Because all real batteries have some internal resistance, when a battery is supplying current to a circuit, the terminal voltage of a real battery is less than the emf.
        - The symbol for the internal resistance of a real battery is typically, r.
    - One way to illustrate a real battery in an electric circuit is shown in yellow.




    - $\Delta V_t = \varepsilon - \Delta V_r \Rightarrow \Delta V_t = \varepsilon - Ir$
    - As current increases, the terminal voltage decreases.
    - The only way to get the terminal voltage to be equal to the emf is to have no current flowing through the battery.

Flipping Physics Lecture Notes:
Series and Parallel Circuits
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-series-parallel-circuits.html




When an anthropomorphic[1] charge has no choice but to go through two circuit elements, those two circuit elements are in *series*. For example, a charge which goes through resistor 1 has no choice but to also go through resistor 2. There is no other path for the anthropomorphic charge to choose.

The currents through the three circuit elements must all be equal:

$I_t = I_1 = I_2$

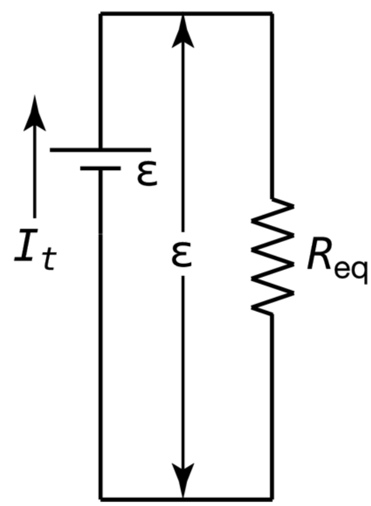
The "t" in the subscript refers to the current at the terminals of the battery which is the current delivered by the battery to the circuit.

The electric potential difference across the battery equals the summation of the electric potential difference across the two resistors:

$$\Delta V_{\text{bottom wire}\rightarrow\text{top wire}} = \varepsilon = \Delta V_1 + \Delta V_2$$

(If you'd prefer to look at this in terms of the electric potential difference around the loop in the circuit:)

$$\Delta V_{\text{loop}} = V_f - V_i = V_a - V_a = 0 = \varepsilon - \Delta V_1 - \Delta V_2 \Rightarrow \varepsilon = \Delta V_1 + \Delta V_2$$




We know Ohm's law: $\Delta V = IR$; therefore, …

$$\Rightarrow \varepsilon = I_t R_{eq} = I_1 R_1 + I_2 R_2$$

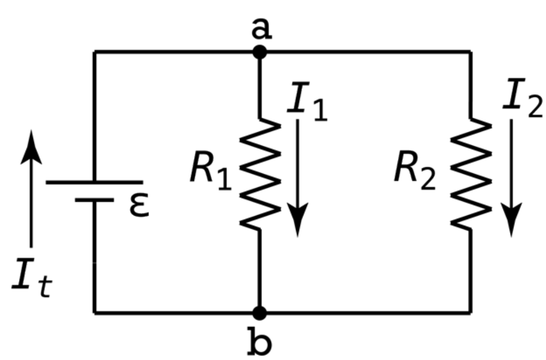
$$\Rightarrow R_{eq} = R_1 + R_2$$

The "eq" in the subscript means equivalent. In other words, $R_{eq}$ is one resistor with the equivalent resistance of the two resistors.

Therefore, the equation for the equivalent resistance of n resistors in series is:

$$R_{\text{eq series}} = \sum_n R_n = R_1 + R_2 + \ldots$$

When an anthropomorphic charge has the choice between two circuit elements and then the paths through those two circuit elements reconverge without going through another circuit element, the two circuit elements are in *parallel*.




When circuit elements are in parallel, their electric potential differences are equal:

$$\varepsilon = \Delta V_1 = \Delta V_2$$

Note the junctions at points a and b. Due to conservation of charge, the net current going into a junction equals the net current coming out of a junction. For junction a:

[1] *Anthropomorphism:* Giving human characteristics or behaviors to non-human objects.

$$I_{\text{in}} = I_{\text{out}} \Rightarrow I_t = I_1 + I_2$$

We can then use Ohm's law:

$$\Delta V = IR \Rightarrow I = \frac{\Delta V}{R} \Rightarrow \frac{\varepsilon}{R_{\text{eq}}} = \frac{\Delta V_1}{R_1} + \frac{\Delta V_2}{R_2} \Rightarrow \frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2}$$

And we get the equivalent resistance for the two resistors in parallel:

$$\Rightarrow R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}$$

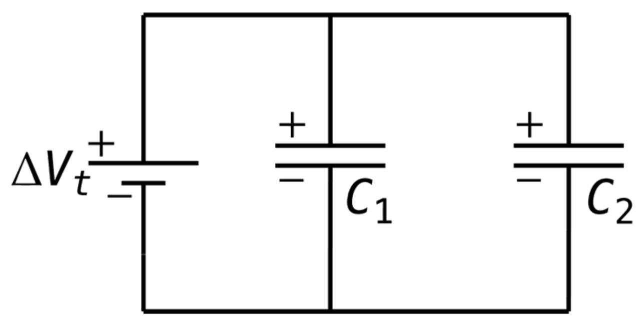
And the equivalent resistance for n resistors in parallel:

$$\Rightarrow R_{\text{eq parallel}} = \left(\sum_n \frac{1}{R_n}\right)^{-1} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \dots\right)^{-1}$$

When we add a resistor in series, the equivalent resistance increases.
When we add a resistor in parallel, the equivalent resistance decreases.

Now let's look at two capacitors in parallel:
We know the electric potential differences are all equal.

$$\Delta V_t = \Delta V_1 = \Delta V_2$$




Because the charges moved to the top plates of the capacitors need to go to either capacitor 1 or capacitor 2, the charge moved by the battery to the plates of the capacitors equals the sum of the charges on the capacitors:

$$Q_t = Q_1 + Q_2$$

We can then use the definition of capacitance:

$$C = \frac{Q}{\Delta V} \Rightarrow Q = C\Delta V$$

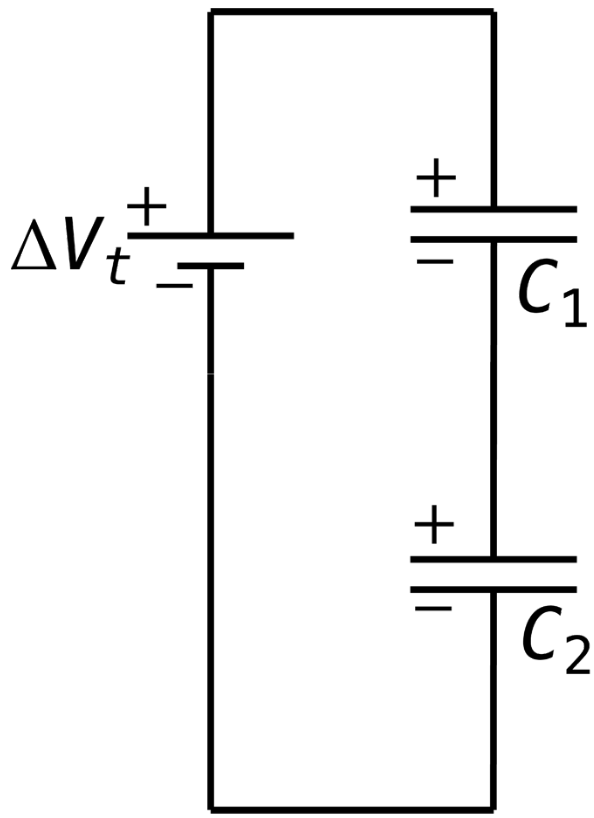
To derive the equivalent capacitance of two capacitors in parallel:

$$\Rightarrow C_{\text{eq}}\Delta V_t = C_1\Delta V_1 + C_2\Delta V_2 \Rightarrow C_{\text{eq}} = C_1 + C_2$$

And the equivalent capacitance of n capacitors in parallel:

$$\Rightarrow C_{\text{eq parallel}} = \sum_n C_n = C_1 + C_2 + \dots$$

And we can now look at two capacitors in series:




The electric potential is the same as resistors in series:

$$\Delta V_t = \Delta V_1 + \Delta V_2$$

And the charges on each capacitor are equal:

$$Q_t = Q_1 = Q_2$$

This is because the magnitude of the charge moved by the battery to the top plate of capacitor 1 and the bottom plate of capacitor 2 are equal in magnitude. And those plates polarize the charges on the wire between the two capacitors and the bottom of capacitor 1 and the top of capacitor 2. This causes all four plates of the two capacitors to have equal magnitude charges. This is an illustration of conservation of charge.

And we can solve for electric potential difference in terms of capacitance and charge:

$$Q = C\Delta V \Rightarrow \Delta V = \frac{Q}{C}$$

And use that to solve for the equivalent capacitance of two capacitors:

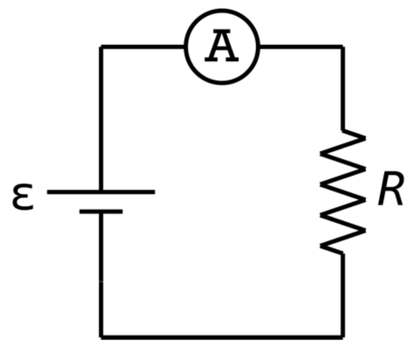
$$\Rightarrow \frac{Q_t}{C_{\text{eq}}} = \frac{Q_1}{C_2} + \frac{Q_2}{C_2} \Rightarrow \frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \Rightarrow C_{\text{eq}} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$

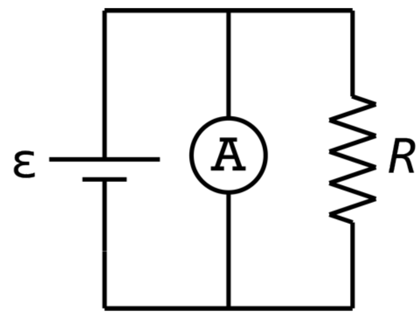
And the equivalent capacitance of n capacitors:

$$\Rightarrow C_{\text{eq series}} = \left(\sum_n \frac{1}{C_n}\right)^{-1} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots\right)^{-1}$$

Notice the equations for resistors and capacitors are reversed. That means that:
When we add a capacitor in parallel, the equivalent capacitance increases.
When we add a capacitor in series, the equivalent capacitance decreases.

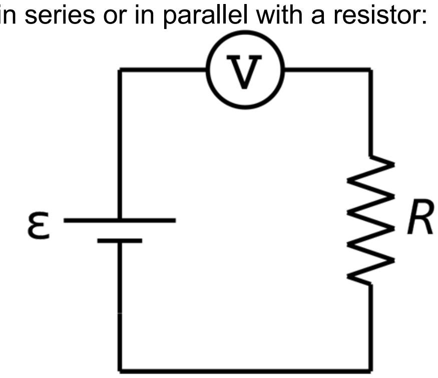
Let’s discuss how to use the tools which measure current and electric potential difference. Starting with the ammeter which measures current or amperes. We need to decide if an ammeter needs to be put in series or parallel with the circuit element it is meant to measure the current through. So, let’s look at what happens when we attempt to measure the current through a resistor using an ammeter in series and in parallel with a resistor:

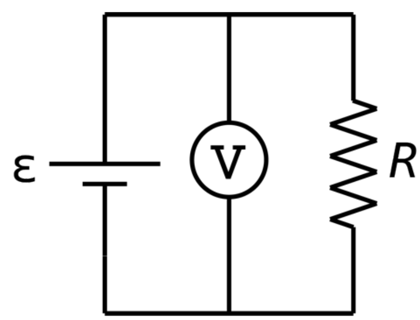





Hopefully you recognize that placing an ammeter in parallel with a resistor will not measure the current through the resistor because the current through the ammeter and the resistor are not the same. Therefore, an ammeter needs to be placed in series with a circuit element to measure the current through that circuit element. Also, the resistance of an ammeter needs to be *very* small. In the above example, if the resistance of the ammeter is not *very* small, it will increase the equivalent resistance of the circuit and decrease the current through the resistor you are trying to measure the current through. Unless otherwise indicated, ammeters in this class are considered to have zero resistance.

And now let's attempt to measure the electric potential difference across a resistor using a voltmeter either in series or in parallel with a resistor:






Hopefully you recognize that placing a voltmeter in series with a resistor will not measure the electric potential difference across the resistor because the voltage across the voltmeter and the resistor are not the same. Therefore, a voltmeter needs to be placed in parallel with a circuit element to measure the voltage across that circuit element. Also, the resistance of a voltmeter needs to be *very* large. In the above example, if the resistance of the voltmeter is not *very* large, it will decrease the equivalent resistance of the circuit, increase the current delivered by the battery, and change the overall properties of the circuit. Unless otherwise indicated, voltmeters in this class are considered to have infinite resistance.

To review:

| ● Ammeters: | ● Voltmeters: |
|---|---|
| ○ Measure current | ○ Measure electric potential difference |
| ○ Placed in *series* with the circuit element | ○ Placed in *parallel* with circuit element |
| ○ Have nearly *zero* resistance* | ○ Have nearly *infinite* resistance* |

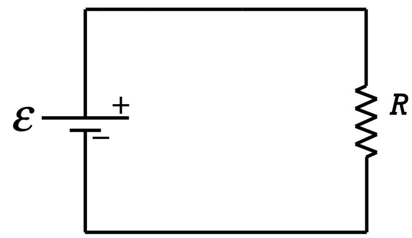
* You may see this called impedance in product literature for Voltmeters and Ammeters, due to the fact that there is more to the behavior of these devices than just resistance. For the purpose of this class and the AP Physics C Electricity and Magnetism exam, it will be called resistance unless otherwise noted.

Flipping Physics Lecture Notes:

Kirchhoff's Rules of Electrical Circuits
https://www.flippingphysics.com/kirchhoff.html

$$\sum_{\substack{\text{closed}\\\text{loop}}} \Delta V = 0$$

Kirchhoff's Two Rules for circuits are very basic rules which are used to understand circuits. Let's start with Kirchhoff's Loop Rule which states that the net electric potential difference around a closed loop equals zero.



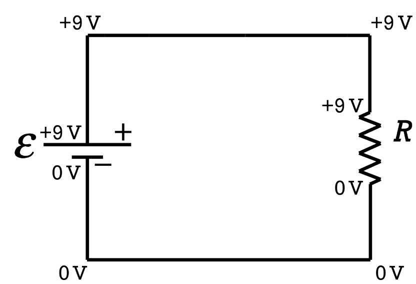

The Loop Rule is essentially conservation of electric potential energy in a circuit. Because electric potential difference equals change in electric potential energy per unit charge, the net change in electric potential energy in a closed loop then equals zero.

$$\sum_{\substack{\text{closed}\\\text{loop}}} \Delta V = 0 \;\&\; \Delta V = \frac{\Delta U_e}{q} \Rightarrow \sum_{\substack{\text{closed}\\\text{loop}}} \frac{\Delta U_e}{q} = 0 \Rightarrow \sum_{\substack{\text{closed}\\\text{loop}}} \Delta U_e = 0$$

Using a gravitational potential energy analogy here, this is like saying, if you drop a mass off a wall, then pick up the mass and return it to its original location, the change in gravitational potential energy of that mass equals zero. We know this to be true because the mass returns back to the same height as where it started, so the mass will have the same gravitational potential energy at the end as it did at the beginning, no matter where we place the horizontal zero line.

Going back to electric potential energy, this means, after a charge goes through one full, closed loop around a circuit, the electric potential energy of the charge will return back to its original value. But because we are using electric potential, we are really talking about the electric potential energy per unit charge at each location.

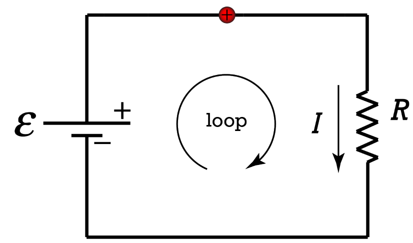
Let's say we have a 9-volt battery. That means we know the electric potential difference across the battery equals 9 volts. As we go from the negative to the positive terminals of the battery, the electric potential will go up. Technically we do not know the electric potential at any point, only the *difference* in the electric potential, however, it is customary to assume the minimum electric potential is zero. That means we are assuming the negative terminal of the battery is at zero volts and the positive terminal of the battery is at positive 9 volts.




Because ideal wires have zero resistance, that means the electric potential in the upper left corner must also be 9 volts, the electric potential in the upper right corner equals 9 volts, and the electric potential at the top of the resistor is 9 volts. Also, the electric potential in the lower left corner must be the same as the negative terminal of the battery, so electric potential in the lower left corner equals 0 volts. Therefore, electric potential in the lower right corner is 0 volts, and the electric potential at the bottom of the resistor equals 0 volts. This means the electric potential difference across the resistor also has a magnitude of 9 volts. In other words, in this circuit with two circuit elements, the two elements, the battery and the resistor, both have the same magnitude electric potential difference.

In a previous lesson we determined that a positive charge in the circuit would be repelled from the positive terminal of the battery and attracted to the negative terminal of the battery, therefore the current in this circuit is clockwise. This means the current is down through the resistor.

There is only one closed loop in our present circuit, so it might not seem obvious that we need to do this, however, we need to define a loop direction. Often the loop direction is the same as the direction which goes from the negative terminal to the positive terminal of the battery and through the battery, therefore, our loop direction for this circuit is clockwise.

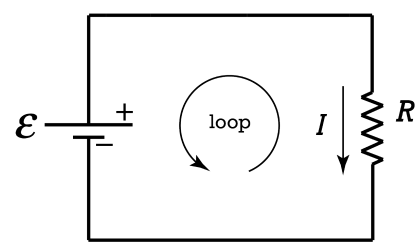



This means as we go in the direction of the loop across the battery, the electric potential goes up because we go from the negative to the positive terminal of the battery. Therefore, when we sum the electric potential differences in our Kirchhoff's loop equation, the electric potential difference across the battery is positive. When we go in the direction of the loop across the resistor, as we illustrated before, the electric potential goes down. Therefore, in our loop equation, the electric potential difference across the resistor is negative. We know the electric potential difference across the battery equals the electromotive force or the emf of the battery. And the electric potential difference across the resistor equals current times resistance. Therefore, we can determine the current in the circuit in terms of the emf of the battery and the resistance of the resistor.

$$\Delta V_{Battery} = \varepsilon \ \& \ \Delta V_{Resistor} = IR$$

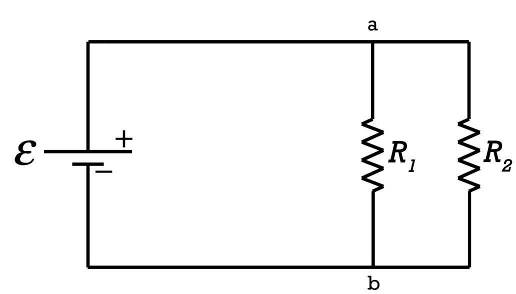
$$\Rightarrow \sum_{\substack{\text{closed}\\ \text{loop}}} \Delta V = \Delta V_{battery} - \Delta V_{Resistor} = 0 = \varepsilon - IR \Rightarrow \varepsilon = IR \Rightarrow I = \frac{\varepsilon}{R}$$

If we had chosen counterclockwise as the loop direction, all of our electric potential differences in Kirchhoff's Loop Rule would have been reversed. Because the loop direction goes from the positive to the negative terminals of the battery, the electric potential difference across the battery is negative, because the electric potential is going down. Because the loop direction through the resistor is opposite the direction of the current direction we defined through the resistor, the electric potential goes up through the resistor and the electric potential difference the across resistor is positive.



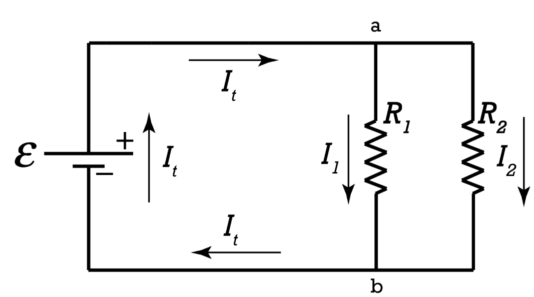

$$\Rightarrow \sum_{\substack{\text{closed}\\ \text{loop}}} \Delta V = -\Delta V_{battery} + \Delta V_{Resistor} = 0 = -\varepsilon + IR \Rightarrow \varepsilon = IR \Rightarrow I = \frac{\varepsilon}{R}$$

Realize, we get the same result for the current in the circuit regardless of which loop direction we choose. If we had chosen an incorrect direction for current, the current ends up being negative, which tells you that you chose the incorrect current direction.




Now let's add a resistor to the circuit and talk about Kirchhoff's Junction Rule which is the result of conservation of charge in the circuit. The rule is sum of the currents entering a junction must equal the sum of the currents leaving a junction, which is conservation of charge:

$$\sum I_{in} = \sum I_{out}$$




Junctions are locations in circuits where at least three circuit paths meet. That means in our circuit we have two junctions which are labelled *a* and *b*. Therefore, the current going into both of those junctions equals the current coming out of those junctions. This means we need to define current directions. We do this the same way we did before, we place a positive test charge in the circuit and see which direction the Law of Charges defines electric force direction on the charge. This means current will go to the right through the top wire, to the left through the bottom wire, and down through both resistors. Let's label those currents as current 1 through resistor 1, current 2 through resistor 2, and current t through the battery because it is the current through the terminals of the battery.

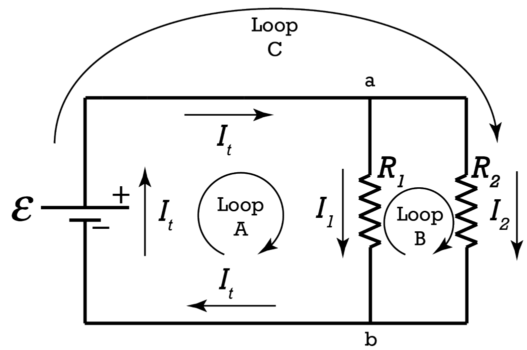
Kirchhoff's Junction Rule equations for this circuit are for:

junction a: $I_t = I_1 + I_2$

junction b: $I_t = I_1 + I_2$

Yes, these two equations are actually the same.

But how do we know *a* and *b* are junctions and the four exterior "corners" of the circuit are not junctions? I know it may seem obvious because there are not at least three circuit paths at any of those locations, however, this is a simple circuit. Again, we return back to placing a positive test charge on the wire. Notice that a charge which approaches point a could go in the wire leading to resistor 1 or in the wire leading to resistor 2. Because junctions are defined as having three circuit paths, any time a charge comes to a fork in the wire, the charge could go down either wire, that makes it a junction. When a charge enters a corner, there is no other choice but to continue along the same wire, therefore none of the corners are junctions.




Let's identify the loops in this second circuit and determine their Kirchhoff's Loop Rule equations. We can define the first loop as the same as the previous circuit, but let's call it loop A with a clockwise direction. There is another loop that contains resistor 1 and resistor 2. Let's call that loop B and also have that be clockwise. Lastly there is a loop all the way around the outside; It includes the battery and resistor 2. Let's call that loop C and have it also be clockwise.

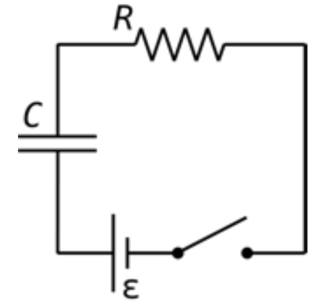
Kirchhoff's Loop Rule equations look like this:

$$\sum_{Loop\ A} \Delta V = \Delta V_t - \Delta V_{R_1} = \varepsilon - I_1 R_1 = 0 \Rightarrow \varepsilon = I_1 R_1 \Rightarrow I_1 = \frac{\varepsilon}{R_1}$$

$$\sum_{Loop\ C} \Delta V = \Delta V_t - \Delta V_{R_2} = \varepsilon - I_2 R_2 = 0 \Rightarrow \varepsilon = I_2 R_2 \Rightarrow I_2 = \frac{\varepsilon}{R_2}$$

$$\sum_{Loop\ B} \Delta V = \Delta V_{R_1} - \Delta V_{R_2} = I_1 R_1 - I_2 R_2 = 0 \Rightarrow I_1 R_1 = I_2 R_2$$

But notice the third equation is actually just a combination of the previous two: $\varepsilon = I_1 R = I_2 R_2$

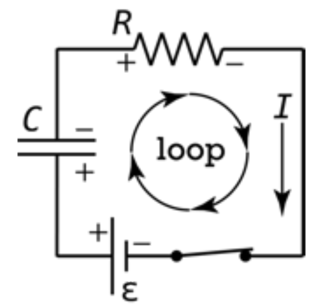



Up until this point we have assumed all changes in electric current, electric potential difference, and charge on capacitor plates were instantaneous. Today, we put a resistor and a capacitor together and learn how those variables change as a function of time. This is called an RC circuit. We start with a circuit composed of an uncharged capacitor, a resistor, a battery, and an open switch, all connected in series.

At time initial, $t_i = 0$, we close the switch.
We are *charging a capacitor through a resistor.*

Let's start by adding a loop in the direction of current flow in the circuit. Then use Kirchhoff's Loop Rule starting in the lower right-hand corner of the circuit:

$$\Delta V_{\text{loop}} = 0 = +\varepsilon - \Delta V_C - \Delta V_R$$




We can use the definition of capacitance to solve for the electric potential difference across the capacitor:

$$C = \frac{Q}{\Delta V} \Rightarrow \Delta V_C = \frac{Q}{C}$$

And we know Ohm's law: $\Delta V_R = IR$

$$\Rightarrow \Delta V_{\text{loop}} = 0 = \varepsilon - \frac{q}{C} - iR$$

Notice we are using lowercase "q" for charge because the charge is changing as a function of time. I wish we had a similar notation for current, I, however, if we used lowercase "i", I am sure it would be more confusing. So, please realize charge, q, and current, I, are both changing as a function of time in the above equation.

Now let's look at limits, starting with $t_i = 0$:

- The initial charge on the capacitor is zero: $q_i = 0$
- This means the initial electric potential difference across the capacitor is also zero:

$$\Rightarrow \Delta V_{C_i} = \frac{q}{C} = \frac{0}{C} = 0$$

- We can now use the loop equation to solve for the initial current through the circuit.

$$\Rightarrow 0 = \varepsilon - \frac{0}{C} - i_{\text{initial}}R \Rightarrow i_{\text{initial}}R = \varepsilon \Rightarrow i_{\text{initial}} = \frac{\varepsilon}{R} = i_{\max}$$

- Because the charge on the capacitor will increase as a function of time, electric potential difference across the capacitor will also increase. This means the current in the circuit will decrease. In other words, the initial current in the circuit is also the maximum current.

And now the limit of "after a long time" or the $t_f \approx \infty$.

- The final current in the circuit is zero: $i_{\text{final}} \approx 0$

- This means the final electric potential difference across the resistor is also zero:

$$\Rightarrow \Delta V_{R_f} = i_{\text{final}} R = (0)R = 0$$

- And we can use the loop equation to solve for the final charge on the capacitor:

$$\Rightarrow 0 = \varepsilon - \frac{q_f}{C} - (0)R \Rightarrow q_f = \varepsilon C = q_{\max}$$

- Because we know the charge has been increasing this whole time, we know this is the maximum charge on the capacitor.

Now let's figure out what happens between $t_i = 0$ and $t_f \approx \infty$, however, before we do, I want to point out that AP Physics C: Electricity and Magnetism students are responsible for knowing how to derive these equations. So, yes, you do need to understand these derivations and be able to do them on your own. And here we go … starting with our Kirchhoff's Loop Rule equation:

$$0 = \varepsilon - \frac{q}{C} - iR \Rightarrow iR = \varepsilon - \frac{q}{C} \Rightarrow i = \frac{\varepsilon}{R} - \frac{q}{RC} \Rightarrow \frac{dq}{dt} = \frac{\varepsilon}{R} - \frac{q}{RC}$$

$$\Rightarrow \frac{dq}{dt} = \frac{1}{RC}(\varepsilon C - q) \Rightarrow \frac{dq}{dt} = -\frac{1}{RC}(q - \varepsilon C)$$

(The above step is the one I find students forget most often. Yes, factor out a negative one on the right-hand side of the equation. Write it down. Remember it. No, it is not an obvious step you need to take.)

$$\Rightarrow \left(\frac{1}{q - \varepsilon C}\right) dq = -\frac{1}{RC} dt \Rightarrow \int_0^q \left(\frac{1}{q - \varepsilon C}\right) dq = -\int_0^t \left(\frac{1}{RC}\right) dt = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow [\ln(q - \varepsilon C)]_0^q = [\ln(q - \varepsilon C)] - [\ln(0 - \varepsilon C)] = \ln\left(\frac{q - \varepsilon C}{-\varepsilon C}\right) = -\frac{t}{RC}$$

$$\Rightarrow e^{\left[\ln\left(\frac{q-\varepsilon C}{-\varepsilon C}\right)\right]} = e^{-\frac{t}{RC}} \Rightarrow \frac{q - \varepsilon C}{-\varepsilon C} = e^{-\frac{t}{RC}} \Rightarrow q - \varepsilon C = (-\varepsilon C)\, e^{-\frac{t}{RC}}$$

$$\Rightarrow q = \varepsilon C - \varepsilon C e^{-\frac{t}{RC}} \Rightarrow q(t) = \varepsilon C\left(1 - e^{-\frac{t}{RC}}\right) \Rightarrow q(t) = q_{\max}\left(1 - e^{-\frac{t}{RC}}\right)$$

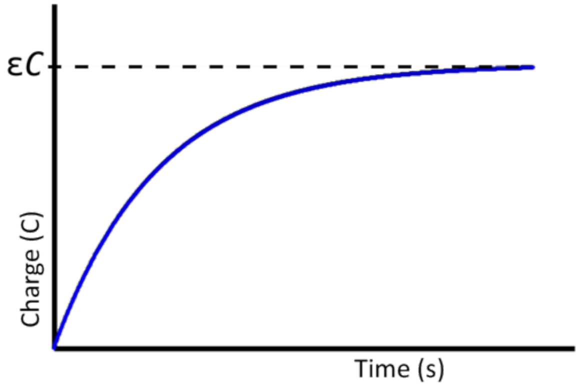
Applicable known equations:

$$\ln x - \ln y = \ln\left(\frac{x}{y}\right) \;\;\&\;\; e^{\ln x} = x$$

Notice this equation fits our limits for charge:

$$q(0) = \varepsilon C\left(1 - e^{-\frac{0}{RC}}\right) = \varepsilon C\left(1 - e^{0}\right) = \varepsilon C(1 - 1) = 0$$

$$q(\infty) = \varepsilon C\left(1 - e^{-\frac{\infty}{RC}}\right) = \varepsilon C\left(1 - e^{-\infty}\right) = \varepsilon C(1 - 0) = \varepsilon C = q_{\max}$$



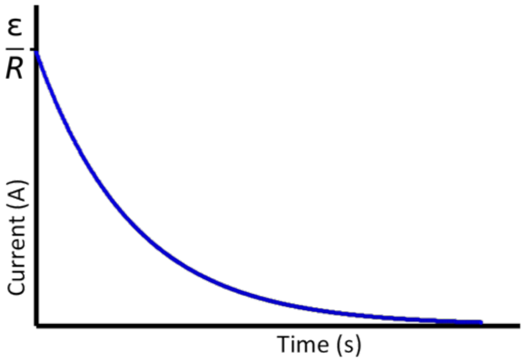

And we can derive the current through the circuit as a function of time:

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(\varepsilon C - \varepsilon C e^{-\frac{t}{RC}}\right) = \frac{d}{dt}\left(-\varepsilon C e^{-\frac{t}{RC}}\right) = -\varepsilon C \frac{d}{dt}\left(e^{-\frac{t}{RC}}\right) = -\varepsilon C\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}}$$

$$\Rightarrow i(t) = \left(\frac{\varepsilon}{R}\right)e^{-\frac{t}{RC}} \Rightarrow i(t) = i_{\max}e^{-\frac{t}{RC}}$$

Again, this fits our limits for current:

$$\Rightarrow i(0) = \left(\frac{\varepsilon}{R}\right)e^{-\frac{0}{RC}} = \frac{\varepsilon}{R} = i_{\max}$$ & $$\Rightarrow i(\infty) = \left(\frac{\varepsilon}{R}\right)e^{-\frac{\infty}{RC}} = 0$$




And now we get to talk about the time constant!

In the equations for charge and current as functions of time, there appears this expression: $e^{-\frac{t}{RC}}$

The time constant equals whatever appears in the denominator of that fraction. In other words, for an RC circuit, the time constant equals resistance times capacitance. The symbol for the time constant is the lowercase Greek letter tau, $\tau$: $\tau = RC$

Before we discuss further what the times constant is, let's determine its units:

$$\tau = RC \Rightarrow \Omega F = \left(\frac{V}{A}\right)\left(\frac{C}{V}\right) = \frac{C}{A} = \frac{C}{\frac{C}{s}} = \frac{1}{\frac{1}{s}} = s$$

$$R = \frac{\Delta V}{I} \Rightarrow \Omega = \frac{V}{A} \ \& \ C = \frac{Q}{\Delta V} \Rightarrow F = \frac{C}{V} \ \& \ I = \frac{dq}{dt} \Rightarrow A = \frac{C}{s}$$

The units for the time constant are seconds; it is the *time* constant.

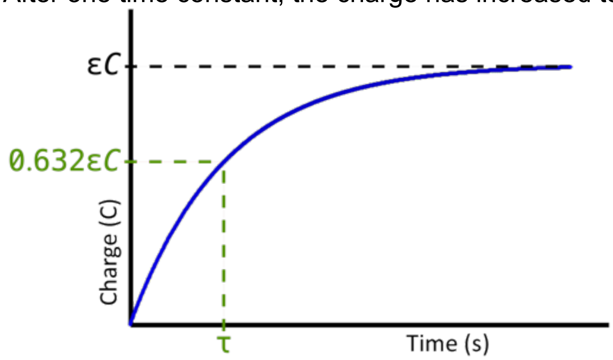
Let's replace RC with the time constant in our charge equation:

$$q(t) = q_{max}\left(1 - e^{-\frac{t}{RC}}\right) \Rightarrow q(t) = q_{max}\left(1 - e^{-\frac{t}{\tau}}\right)$$

And determine the charge on the capacitor after one time constant:

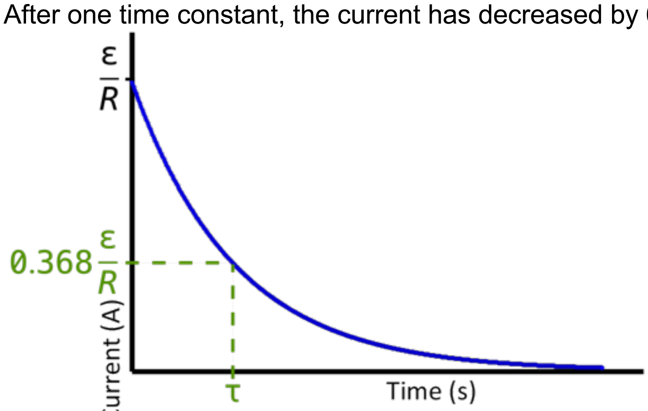
$$q(\tau) = q_{max}\left(1 - e^{-\frac{\tau}{\tau}}\right) = q_{max}\left(1 - e^{-1}\right) = q_{max}(1 - 0.368) \Rightarrow q(\tau) = 0.632 q_{max}$$

After one time constant, the charge has increased to 63.2% of its maximum value.




$$i(t) = i_{max} e^{-\frac{t}{\tau}} \Rightarrow i(\tau) = i_{max} e^{-\frac{\tau}{\tau}} = i_{max} e^{-1} \Rightarrow i(\tau) = 0.368 i_{max}$$

After one time constant, the current has decreased by 63.2% from its maximum value.




The time constant is the time it takes for a change of 63.2%. If you want to know more about the time constant, I talk about it in more detail in my video *Time Constant and the Drag Force*: https://www.flippingphysics.com/drag-force-time-constant.html

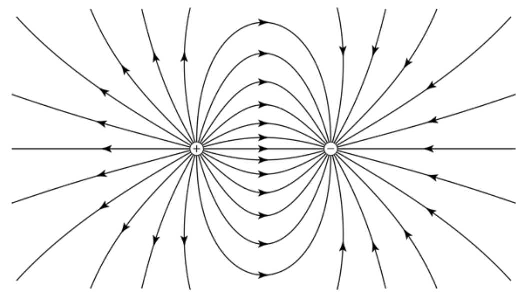
There are similar equations for discharging a capacitor through a resistor which we are not going to derive today.

Please realize the following two calculus equations are on the AP Equation Sheet:

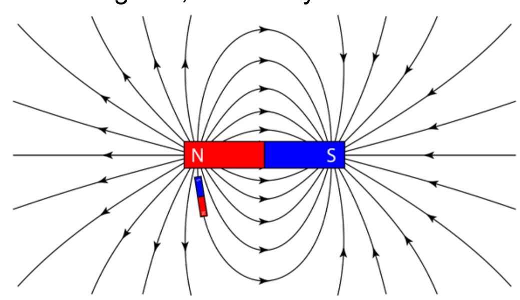
$$\int \frac{dx}{x + a} = \ln|x + a| \ \& \ \frac{d}{dx}\left(e^{ax}\right) = a e^{ax}$$

Magnetic fields (or “B” fields) are be created by magnetic dipoles:

- Just like electric charges are described as positive and negative charges, magnetic poles are described as north and south poles.
    - A magnetic monopole has never been found.
        - This does not mean magnetic monopoles do not exist.
            - We cannot prove magnetic monopoles do not exist.
                - We can only say we have no evidence that they exist.
    - If a magnetic dipole is broken in half, it becomes two new magnetic dipoles.
- Like poles repel and unlike poles attract.
    - Just like the Law of Charges



- The magnetic field caused by a magnetic dipole looks remarkably like the electric field caused by an electric dipole.
    - B field lines external to the magnet, point from north pole to south pole.
        - Just like E field points from positive charge to negative charge.
    - Magnetic field lines must be closed loops.
        - Due to Gauss’ law for magnetism which we will get to, eventually.
        - This means B fields inside the magnet point from the south pole to the north pole, to complete the closed loop.
    - A magnetic dipole placed in a magnetic field will align itself with the magnetic field.
        - Think *compass!*




- For planet Earth:
    - The location of the geographic north pole is close to that of the magnetic south pole.
    - The location of the geographic south pole is close to that of the magnetic north pole.
        - The north pole of a compass points north because it is attracted to the magnetic south pole of the Earth.
            - (unlike poles attract)
    - The magnetic field of the Earth can be approximated as a magnetic dipole.

Magnetic dipoles are the result of electric charges moving in circles.

- We will cover electric charges moving in circles creating magnetic fields extensively later. At this point, just know that electric charges moving in circles create magnetic fields.
- The magnetism of magnets is most often the motion of electrons moving in circles inside them.
- Permanent magnetic dipoles and induced (temporary) magnetic dipoles are a property of the object which results from the alignment of magnetic dipoles within the object.

The material composition of a magnet affects its magnetic behavior when it is placed in an external magnetic field:

- *Ferromagnetic* materials can be *permanently* magnetized by an external magnetic field.
    - The alignment of the magnetic domains or atomic magnetic dipoles is *permanent*.
    - Example materials: nickel, iron, cobalt
- *Paramagnetic* materials are only *temporarily* magnetized by an external magnetic field.
    - The alignment of the magnetic domains or atomic magnetic dipoles is *temporary*.
    - Example materials: aluminum, magnesium, titanium

Just like materials have an electric permittivity, ε, materials also have a magnetic permeability, μ:

- Magnetic permeability: the measurement of the amount of magnetization a material has in response to an external magnetic field.
    - Ferromagnetic materials have high magnetic permeabilities that increase in the presence of an external magnetic field.
    - Paramagnetic materials have low magnetic permeabilities.
    - The magnetic permeability of materials is not constant. It changes depending on various factors such as temperature, orientation, and the strength of the external magnetic field.
- The magnetic permeability of free space has a constant value, $\mu_0$: $\mu_0 = 4\pi \times 10^{-7} \frac{T \cdot m}{A}$

A magnetic field is defined by the fact that a moving electric charge in a B field can experience a magnetic force, $F_B$.

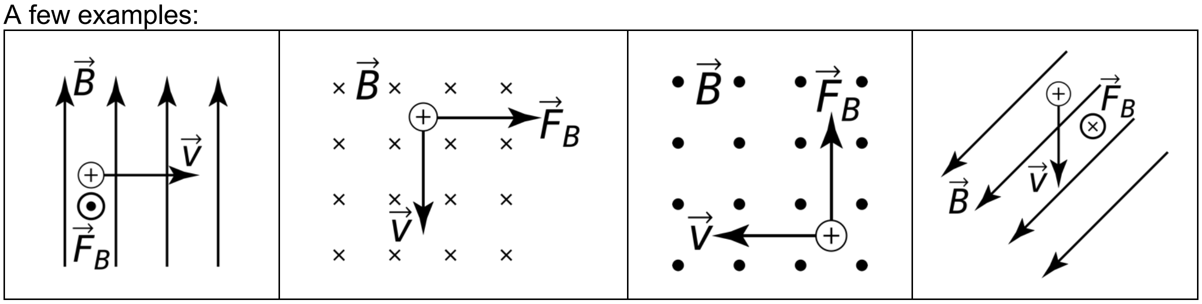
- $\vec{F}_B = q\vec{v} \times \vec{B} \Rightarrow \|F_B\| = qvB\sin\theta$
    - This equation is an experimentally determined equation. In other words, there is no way to mathematically derive it! We know it is true because we have repeatedly measured it.
    - Notice the similarities to the torque equations: $\vec{\tau} = \vec{r} \times \vec{F} \Rightarrow \|\tau\| = rF\sin\theta$
- $\Rightarrow B = \frac{F_B}{qv\sin\theta} \Rightarrow \frac{N}{C\left(\frac{m}{s}\right)} = \frac{N}{\left(\frac{C}{s}\right)m} = \frac{N}{A \cdot m} = \text{tesla}, T$
    - $1 \text{ tesla}, T = 10{,}000 \text{ gauss}, G$

Please recognize that the magnetic field is a vector. To that end, we need to know the direction of the magnetic force acting on an electric charge moving in a magnetic field. For that we use …

*The Right-Hand Rule:* [Don't be too cool. Limber up. Find your right hand.]

- Fingers point in the direction of the electric charge velocity.
- Fingers curl in the direction of the magnetic field.
    - It's a good rule of thumb[1] to start at 90°.
- Thumb points in the direction of the magnetic force on a positive charge.
    - For a negative charge, the thumb points 180° from the direction of the magnetic force.
    - Make sure your thumb points normal to the plane created by the velocity of the electric charge and the magnetic field.
        - In other words, realize the direction of the magnetic force is always normal to the plane created by the velocity of the electric charge and the magnetic field.
- Realize, the cross-product version of the magnetic force equation also gives you the direction of the magnetic force in terms of unit vectors.
- Since examples of this concept require vectors in all three dimensions, we introduce two symbols to indicate direction perpendicular to the page. A dot for out of the page, and an X for into the page, like the pointed tip and fletching (feathers) of a flying arrow respectively.

A few examples:

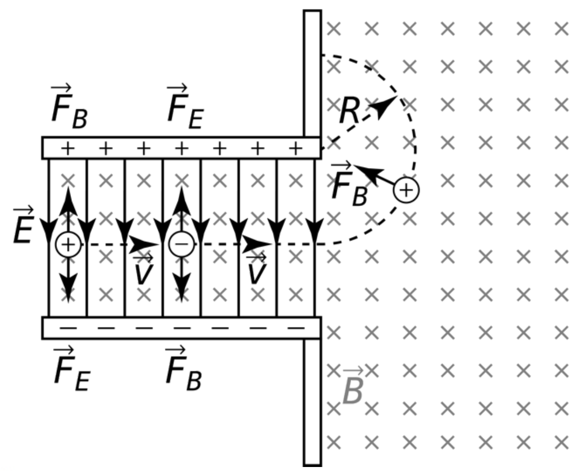



[1] Ha ha ha!

What if we have a series of charges all moving in the same direction? Like a current carrying wire?

- We already derived the equation for the current in a wire: $I = nAv_dq$
- And we know the magnetic force acting on *each individual charge* moving in the wire:
  $\vec{F}_B = q\vec{v} \times \vec{B}$
- However, what we want to know is the net magnetic force acting on all the charges moving in the wire. So, we need to use charge carrier density, n:
  - $$n = \frac{\text{\# of charges}}{V} \Rightarrow \text{\# of charges} = nV = nAL$$
- Which we can use to get the magnetic force acting on *all the charges* moving in the wire:
  $$\vec{F}_B = \left(q\vec{v} \times \vec{B}\right) nAL = nAvq\vec{L} \times \vec{B}$$
- And we have derived the general equations for the magnetic force on a current carrying wire both for a straight wire and, using an integral, a wire that does not follow a straight path.
  $$\Rightarrow \vec{F}_B = I\vec{L} \times \vec{B} \Rightarrow \vec{F}_B = \int I\left(d\vec{L} \times \vec{B}\right) \quad \& \quad \|F_B\| = ILB\sin\theta$$
- You use the same wonderful right-hand rule to determine the direction of this force.

Because it involves many concepts that are likely to come up on the AP exam, let's take a moment to analyze a mass spectrometer:




The magnetic field is uniform into the page throughout, and in the velocity selector, the electric field uniform and down.

Velocity Selector:

- For a positive charge the magnetic force is up and the Coulomb force is down.
- For a negative charge the magnetic force is down and the Coulomb force is up.
- Regardless of whether the charge is positive or negative, the free body diagrams result in the same Newton's Second Law equation:

$$\sum F_y = F_B - F_E = ma_y = 0 \Rightarrow F_B = F_E$$

$$\Rightarrow qvB\sin\theta = qE \Rightarrow vB\sin(90^\circ) = E \Rightarrow v = \frac{E}{B}$$

- So, all charged objects with the same constant velocity will all move in a straight horizontal line in the velocity selector. Regardless of mass, charge sign, and charge magnitude.

Deflection Chamber:

- The uniform magnetic field is the only field present in the deflection chamber.
- The only force acting on the charged particle is the magnetic force which acts inward.
  - The charged particles will move along a circular path with radius, R.
  - Positive charges will be deflected upward.
  - Negative charges will be deflected downward.
- Again, we use Newton's Second Law:

$$\sum F_{\text{in}} = F_B = ma_c \Rightarrow qvB\sin\theta = qvB\sin(90^\circ) = qvB = m\left(\frac{v_t^2}{R}\right)$$

$$\Rightarrow qB = \frac{mv_t}{R} \Rightarrow qB = \frac{m\left(\frac{E}{B}\right)}{R} \Rightarrow mE = qRB^2 \Rightarrow \frac{m}{q} = \frac{RB^2}{E}$$

The mass spectrometer is a tool for determining velocities and mass-to-charge ratios of electric charges. Imagine how useful this could be for learning information about new particles!

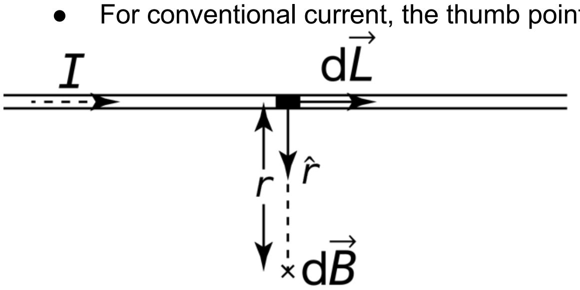
Flipping Physics Lecture Notes:
Biot-Savart and Ampère's Laws
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-biot-savart-ampere.html

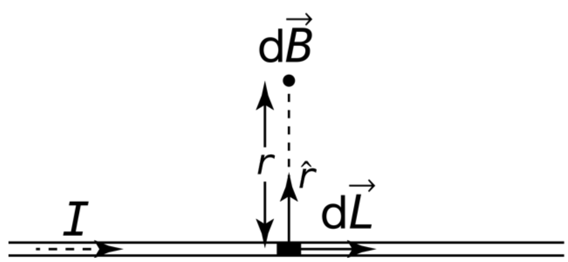
The Biot-Savart law:

- $$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{L} \times \hat{r}}{r^2}$$
- This is an experimentally determined equation; you cannot derive it.
- Unit vector r is a position vector which points from the location of the infinitesimally small length of the wire, dL, to the location of the infinitesimally small magnetic field, dB.
    - r is the magnitude of the distance between those two points
- Magnetic permeability, μ, is the measurement of the amount of magnetization of a material in response to an external magnetic field. $\mu_0$ is the magnetic permeability of free space:
  $$\mu_0 = 4\pi \times 10^{-7}\frac{T \cdot m}{A}$$
- This equation shows that a current carrying wire creates a magnetic field. In fact, because current is composed of individually moving electric charges, even a single moving electric charge causes a magnetic field.

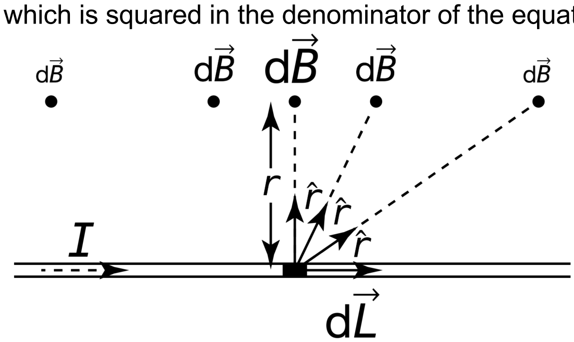
The direction of the magnetic field created by a current carrying wire can be seen using the Biot-Savart law. It is the cross product, so again, we use the right-hand rule!

- Fingers point in the direction of current/wire.
- Fingers curl in the direction of unit vector r.
- For conventional current, the thumb points in the direction of the magnetic field.

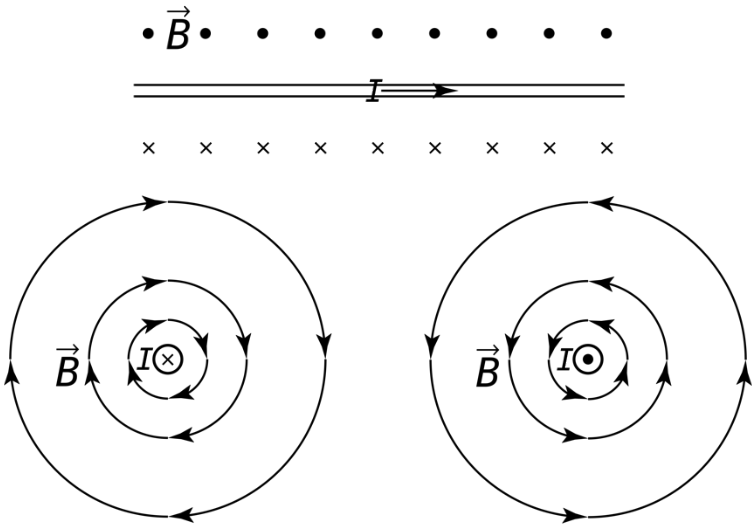



Notice the direction of the magnetic field caused by an infinitesimally small portion of the current carrying wire, dL, is the same along a line parallel to the straight wire, however, the magnitude of the magnetic field decreases as you get farther from a line perpendicular to the straight wire. The direction remains the same because the cross product of dL and unit vector r always gives the same direction. The magnitude decreases as the value of r, which is squared in the denominator of the equation, increases.



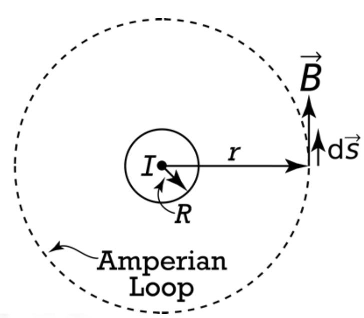

However, now realize that there are, for an infinitely long, straight, current carrying wire, an infinite number of dL's and all of their magnetic fields add up to cause the magnetic field to have a uniform value at a distance r straight out from the wire. And, the magnitude of the magnetic field decreases as r, the distance from the wire, increases.




- An alternate right-hand rule exclusively for the magnetic field which surrounds a current carrying wire is:
    - Point thumb in direction of current.
        - Fingers curl in the direction of the magnetic field.
- The Biot-Savart law can also be used to determine the magnitude of the magnetic field a distance r from an infinitely long, straight, current carrying wire. That equation is: $B = \frac{\mu_0 I}{2\pi r}$
    - We now know the magnetic field magnitude is inversely proportional to distance from the wire.

Next up we have Ampère's law, which is the magnetic field equivalent to Gauss' law:

- Gauss' law: $\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$
    - Closed surface integral and charge inside a Gaussian surface.
- Ampère's law: $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{in}$
    - Closed loop integral and current inside an Amperian loop.



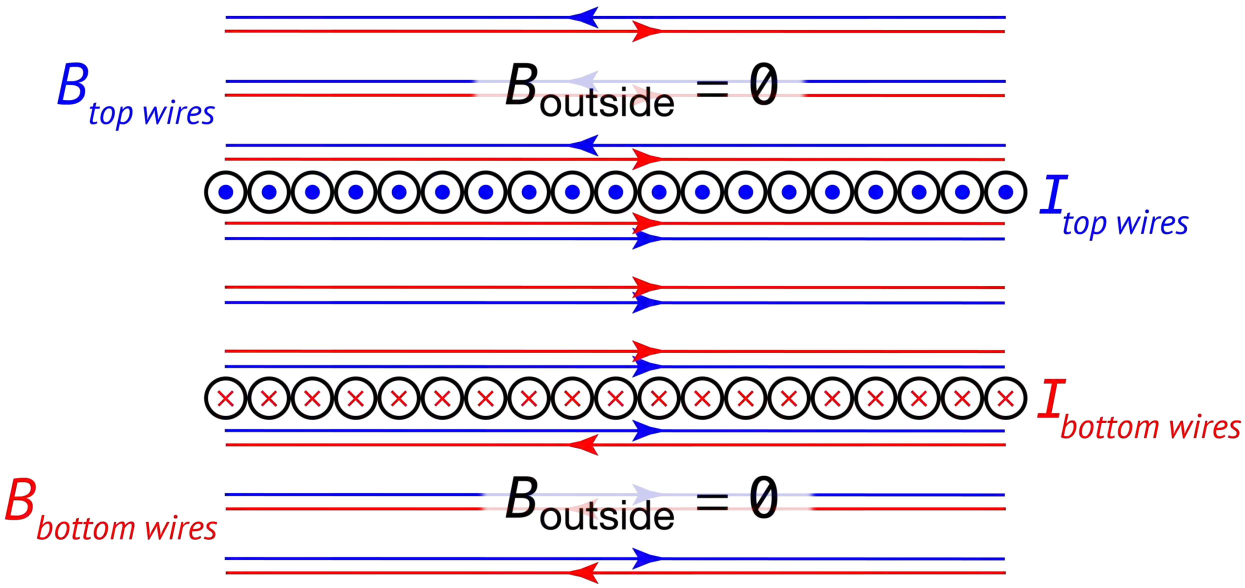

Example: Determine the magnitude of the magnetic field outside an infinitely long, straight, wire with radius R and current I.

Start by drawing an Amperian loop in the shape of a circuit of radius $r \geq R$ which is concentric with the wire. And let's use Ampère's law.

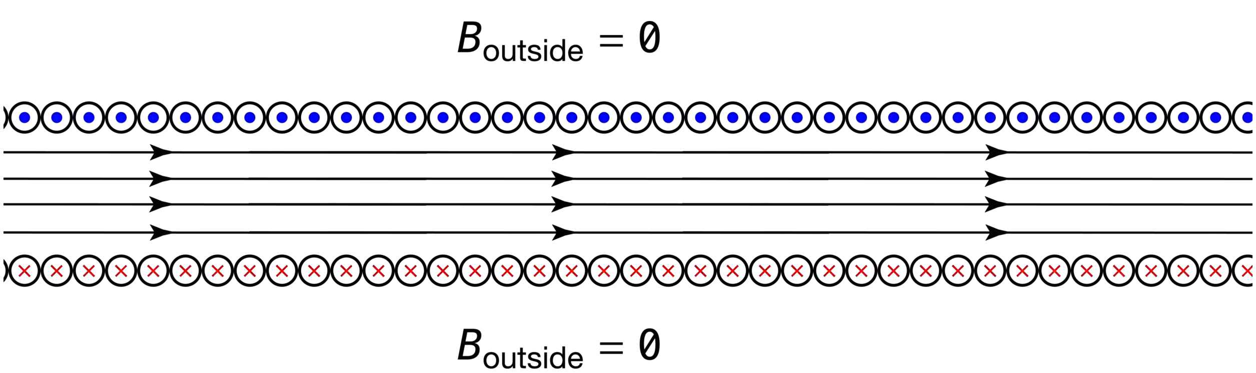
$$\Rightarrow \oint B ds \cos\theta = \oint B ds \cos 0° = B \oint ds = B(2\pi r) = \mu_0 I_{in}$$

$$\Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

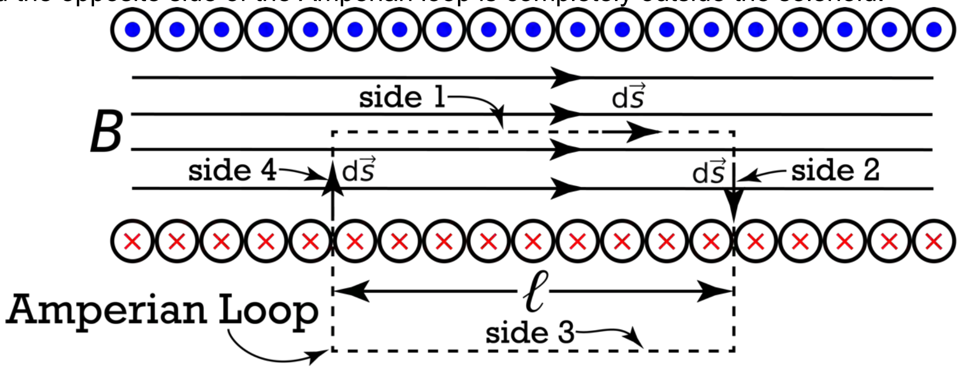
Next let's look at a solenoid. A very common tool for creating a uniform magnetic field. A typical solenoid is a single, very long, current carrying, insulated wire wrapped to form a hollow cylinder. An ideal solenoid has a length which is much, much larger than its diameter. The cross section of a solenoid looks like this.




Outside the solenoid, the magnetic field caused by the current in the top wires completely cancels out the magnetic field caused by the bottom wires. In other words, an ideal solenoid has zero magnetic field outside the cylinder of the solenoid. (ideal solenoid below)



Now let's derive the equation for the magnetic field inside an ideal solenoid. In order to do so, we begin with Ampère's law and draw an Amperian loop. Just like Gaussian surfaces, we want to pick Amperian loops to have sides which are at integer multiples of 90° relative to the magnetic field, and such that the magnetic field is uniform on the sides of the Amperian loop. For an ideal solenoid, we pick an Amperian loop shape of a rectangle with one side inside the solenoid and parallel to the magnetic field inside the solenoid and the opposite side of the Amperian loop is completely outside the solenoid.




And now we can begin using Ampère's law:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{in} \Rightarrow \int_1 \vec{B} \cdot d\vec{s} + \int_2 \vec{B} \cdot d\vec{s} + \int_3 \vec{B} \cdot d\vec{s} + \int_4 \vec{B} \cdot d\vec{s} = \mu_0 I_{in}$$

For side 3, the magnetic field is zero outside the solenoid, so that integral equals zero. For sides 2 and 4, the magnetic field and ds are 90° to one another and the cosine of 90° is zero, so both of those integrals equal zero. That means, the only integral which remains is the integral for side 1.

$$\Rightarrow \int_1 \vec{B} \cdot d\vec{s} = B \int_1 ds \cos 0° = B\ell = \mu_0 N I \;\; \& \;\; I_{in} = NI$$

$$\Rightarrow B = \frac{\mu_0 N I}{\ell} \;\; \& \;\; n = \frac{N}{\ell} \Rightarrow B_{solenoid} = \mu_0 n I$$

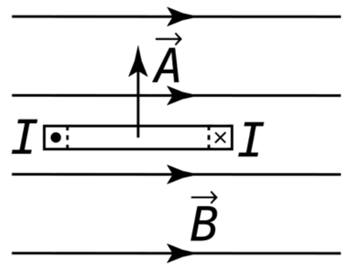
Where "n" is the turn density of the solenoid.

Flipping Physics Lecture Notes:
Electromagnetic Induction
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-electromagnetic-induction.html

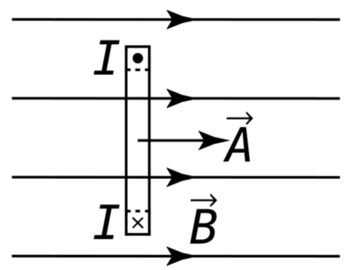
Before we learn about electromagnetic induction, we need to learn about magnetic flux. Before we do that, let's review electric flux:

- Electric flux is the measure of the number of electric field lines which pass through a surface.
- When the electric field is uniform, and the surface is a two-dimensional plane:
  $\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta$
- The general equation for electric flux: $\Phi_E = \int \vec{E} \cdot d\vec{A}$

Magnetic flux:




- Magnetic flux is the measure of the number of magnetic field lines which pass through a surface.
- When the magnetic field is uniform, and the surface is a two-dimensional plane: $\Phi_B = \vec{B} \cdot \vec{A} = BA\cos\theta$
- The general equation for magnetic flux:
  $$\Phi_B = \int \vec{B} \cdot d\vec{A} \Rightarrow T \cdot m^2 = \text{webbers, } Wb$$
- Example #1: Current through a wire loop. Use the right-hand rule to determine the direction of the area vector. (Similar to the right-hand rule for angular velocity direction.) Fingers curl in the direction of the current, thumb points in the direction of the area vector.
  $\Phi_B = BA\cos\theta = BA\cos 90° = 0$
- Example #2: $\Phi_B = BA\cos\theta = BA\cos 0° = \Phi_{B_{max}}$




Gauss's law has to do with electric flux:

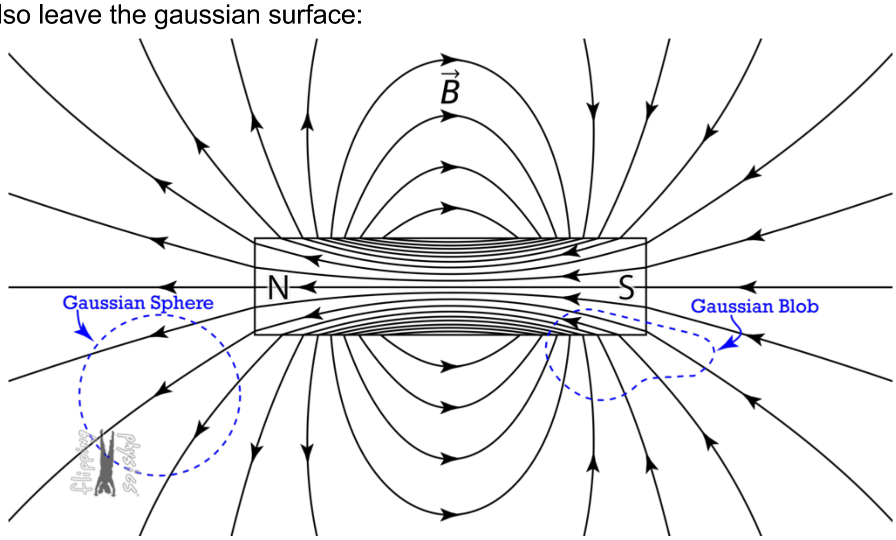
$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$$

*Gauss's law for magnetism* has to do with magnetic flux:

- Because a magnetic monopole has never been found in nature or in a manmade experiment, every magnetic field line is a closed loop.
- Therefore, no matter what shape the gaussian surface has, every magnetic field line which enters the gaussian surface will also leave the gaussian surface:

$$\Phi_B = \oint \vec{B} \cdot d\vec{A} = 0$$

- The above equation, Gauss's law for magnetism, is the second of Maxwell's equations which are a collection of equations which fully describe electromagnetism.



Electromagnetic Induction:

- We have already discussed that moving electric charges create magnetic fields.
- It should be no surprise that moving magnetic poles create electric fields.
  - Notice how these interact with one another!
- When a magnetic field changes over time, this can induce an electric potential difference called an induced emf, this causes charge to flow in a closed loop of wire which is called an induced current. More specifically, the relationship is between a changing magnetic flux and the resulting induced emf in a single closed loop of wire and is described by Faraday's law of electromagnetic induction:

$$|\varepsilon| = \left|\frac{d\Phi_B}{dt}\right|$$

  - Induced emf = the derivative of magnetic flux with respect to time. (magnitudes)
- Substitute in the equation for magnetic flux:

$$|\varepsilon| = N\left|\frac{d\Phi_B}{dt}\right| = N\left|\frac{d\left(\vec{B}\cdot\vec{A}\right)}{dt}\right| = N\left|\frac{d\,(BA\cos\theta)}{dt}\right|$$

  - N is the number of loops
  - An emf can be induced by changing:
    - Magnitude of the magnetic field.
    - Area enclosed by the loop.
    - Angle between magnetic field and loop area. (θ between $\vec{B}$ and $\vec{A}$)
    - Or any combination of the three.
      - In other words, if the only one of those three ($\vec{B}$, $\vec{A}$, and θ) which is changing is the magnitude of the magnetic field, then the magnitude of the induced emf through one loop of wire is:

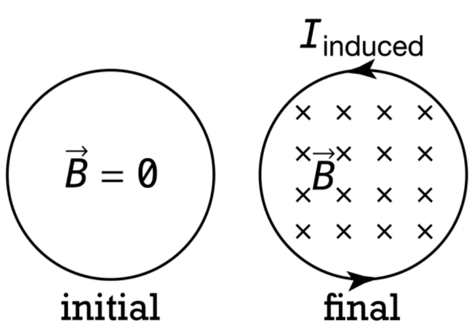
$$|\varepsilon| = \left|A\cos\theta\left(\frac{dB}{dt}\right)\right|$$

- Electromagnetic induction is the process of inducing an electromotive force by a change in magnetic flux.
- Faraday's law is the third of Maxwell's equations which are a collection of equations which fully describe electromagnetism.

Now we need to determine the direction of the induced emf caused by a changing magnetic flux. That is shown by removing the absolute value from the equation, which gives us, assuming only one loop:

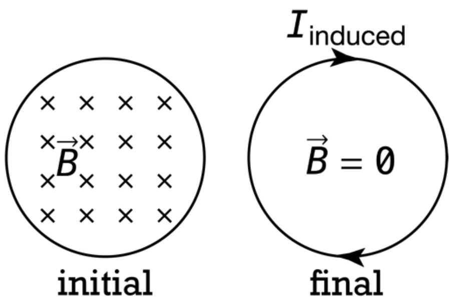
$$\varepsilon = -\frac{d\Phi_B}{dt}$$

- The negative in this equation means the induced emf is opposite the direction of the change in magnetic flux.
- The direction of the induced emf is called Lenz' law.
  - Yes, the negative added to Faraday's law is called Lenz' law.
  - Lenz' law: The current induced in a circuit due to a change in a magnetic field is directed to oppose the change in magnetic flux and to exert a mechanical force which opposes the motion.
- We use the right-hand rule[1] to determine the direction of the induced emf. Examples below:



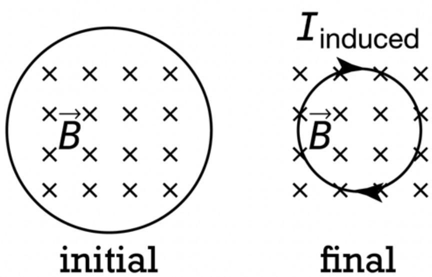

- Zero initial magnetic flux inside the loop.
- Original B field is into the screen and increasing, therefore the original magnetic flux is increasing.
- Induced magnetic field opposes the change in the original magnetic flux and therefore is induced out of the screen to counteract the change in original magnetic flux.
- According to the right-hand rule, fingers curl out of the screen in the direction of the induced magnetic field, thumb points in the direction of the induced current which is counterclockwise.

[1] This is the "alternate" right-hand rule with the thumb pointing in the direction of the current in the wire and fingers curling in the direction of the magnetic field created by the current in the wire.



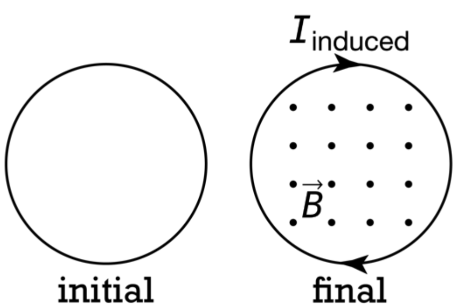

● Original B field in the loop is into the screen and decreasing, which means the original magnetic flux is decreasing.
● $B_{induced}$ opposes this change in magnetic flux and attempts to maintain the original magnetic flux. Therefore, $B_{induced}$ is into the screen.
● According to the right-hand rule, fingers curl into the screen in the direction of the induced magnetic field, thumb points in the direction of the induced current which is clockwise.

Note: Magnetic flux is a dot product, so magnetic flux is a scalar. So, the induced magnetic flux does not have a direction, however, the induced magnetic field does have a direction and the direction of the induced magnetic field in the plane of the loop is always normal to the loop in which the induced current is created.

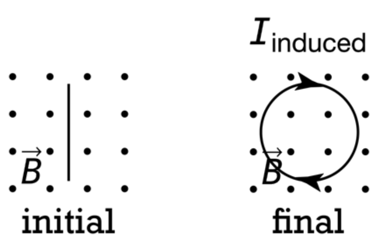



● Original B field inside the loop is into the screen and the area is decreasing which means the original magnetic flux is decreasing.
● $B_{induced}$ opposes this change in magnetic flux and produces a magnetic field to maintain the original magnetic flux. Therefore, $B_{induced}$ is into the screen.
● According to the right-hand rule, fingers curl into the screen in the direction of the induced magnetic field, thumb points in the direction of the induced current which is clockwise.

(The next example was cut out of the video, however, y'all still get to enjoy it here!)

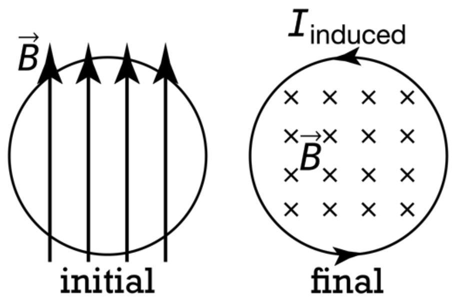



● There is no original B field so no original magnetic flux. The B field is increasing out of the screen so the original magnetic flux is increasing.
● $B_{induced}$ opposes this change in magnetic flux and produces a magnetic field to maintain the original magnetic flux. Therefore, $B_{induced}$ is into the screen.
● According to the right-hand rule, fingers curl into the screen in the direction of the induced magnetic field, thumb points in the direction of the induced current which is counterclockwise.

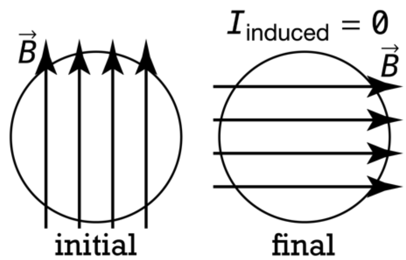



● B field is originally parallel to the loop, so there is zero original magnetic flux through the loop. Loop turns to cause the area of the loop to now be normal to the B field which is out of the screen. So, the original magnetic flux is increasing.
● $B_{induced}$ opposes this change in magnetic flux and produces a magnetic field to maintain the original magnetic flux. Therefore, $B_{induced}$ is into the screen.
● According to the right-hand rule, fingers curl into the screen in the direction of the induced magnetic field, thumb points in the direction of the induced current which is clockwise.

Note: No matter which way the loop is turned, the change in the magnetic flux through the loop is the same and the induced magnetic field is into the screen caused by the induced current which is clockwise from this perspective.




● B field is originally parallel to the loop, so there is no original magnetic flux. B field turns to now be into the screen. So, the original magnetic flux is increasing.
● $B_{\text{induced}}$ opposes this change in magnetic flux and produces a magnetic field to maintain the original magnetic flux. Therefore, $B_{\text{induced}}$ is out of the screen.
● According to the right-hand rule, fingers curl out of the screen in the direction of the induced magnetic field, thumb points in the direction of the induced current which is counterclockwise.




● B field is originally parallel to the loop, so there is no original magnetic flux. B field turns to now be … still parallel to the loop. So, the magnetic flux through the loop is still zero.
● No change in the magnetic flux means there is no induced current. ☺️

We now have covered all four of Maxwell's equations which are a collection of equations which fully describe electromagnetism:

1) Gauss' law: $$\Phi_E = \oint_{\text{surface}} \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$
2) Gauss' law in magnetism: $$\Phi_B = \oint_{\text{surface}} \vec{B} \cdot d\vec{A} = 0$$
3) General form of Maxwell-Faraday's law of induction: $$\varepsilon = \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$
4) The Ampère-Maxwell law: $$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{in}} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

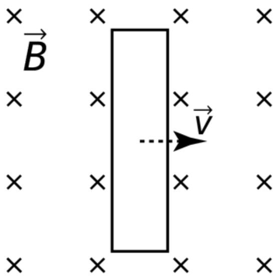
Maxwell's third equation is:

- The Faraday's law of induction we previously learned: $\varepsilon = -\frac{d\Phi_B}{dt}$
  - Which shows that changing magnetic fields create an electric potential difference
- plus the more general addition: $\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$
  - Which shows that a changing magnetic field must also create a nonconservative electric field.

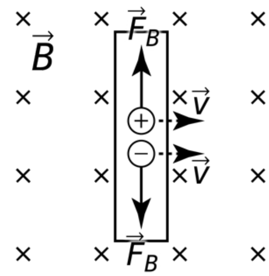
Maxwell's fourth equation is:

- Ampère's law: $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{in}}$
  - Which shows that magnetic fields can be generated by electric currents
- plus Maxwell's addition of $\varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$
  - Which shows that a changing electric field creates a magnetic field.
    - In a similar manner to how a moving charge creates a magnetic field.

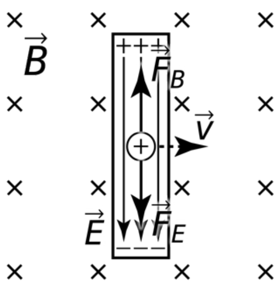
Believe it or not, but our discussion of induced forces begins with two derivations of *motional emf*. Motional emf is the idea that the motion of a conductor moving in a magnetic field can cause charges to move in the conductor creating a voltage across the conductor. In other words, a conductor moving in a magnetic field can acquire an induced emf across it.




- The conductor is moving to the right with a constant velocity at a right angle to a magnetic field which is into the page.
- According to the right-hand rule, positive charges will experience an upward magnetic force, and negative charges will experience a downward magnetic force.
- This will result in the movement of charges with the final result being that there will be a net positive charge on the top end of the conductor and a net negative charge on the bottom end of the conductor. This arrangement of charges creates a uniform, downward electric field in the moving conductor.




- As a result of the downward electric field in the conductor, positive charges will experience a downward electrostatic force, and negative charges will experience an upward electrostatic force.




- Because the conductor is moving at a constant velocity, the charges will arrange themselves such that equilibrium is reached between the magnetic and electric forces acting on the charges such that the electric field has a constant magnitude and the charges in the conductor are moving with a constant velocity to the right; there is no vertical motion of the electric charges.
- We can now sum the forces on a positive charge.
  - The same final equation is derived when using a negative charge.

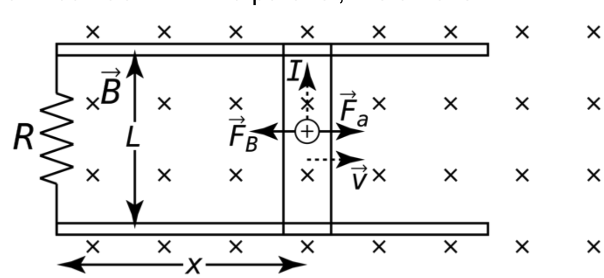
$$\sum \vec{F}_y = F_B - F_E = ma_y = m(0) = 0 \Rightarrow F_B = F_E \Rightarrow qvB\sin\theta = qE$$
$$\Rightarrow vB\sin 90^\circ = E \Rightarrow vB = E$$

Previously we derived the equation relating voltage and a uniform electric field. We have already identified the direction of the electric field, so we only need the absolute value of the voltage.

$$\& \ \Delta V = -Ed \Rightarrow |\Delta V| = EL \Rightarrow E = \frac{\Delta V}{L} = vB \Rightarrow \Delta V = vBL \Rightarrow \varepsilon = vBL$$

L is the length of the conductor. We have derived the voltage or the induced emf across the conductor moving at a right angle to a uniform magnetic field. This is called *motional emf*.

There is actually an entirely different approach to deriving the same motional emf equation. This approach starts with a conductor moving to the right while in contact with two parallel, metal rails connected by a wire at the left end with a uniform magnetic field going into the page. The resistance of the circuit is represented by the resistor shown in the wire on the left. A force is applied to the conductor to cause it to move to the right. We can use Lenz' law to determine the direction of the induced current in the loop.




- The magnetic field is into the screen and the magnetic flux is increasing because the area of the loop is increasing which increases the number of field lines passing through the loop.

- The induced magnetic field opposes this change in flux and is directed out of the page.
- Using the alternate right-hand rule, our fingers curl in the direction of the induced magnetic field which is out of the page inside the loop and our thumb points in the counterclockwise direction which is in the direction of the induced current in the loop.
- Notice this means that, because positive charges are moving in the direction of conventional current in the conductor, we can use the right-hand rule to show that the fingers point in the direction of the motion of the positive charges which is up, fingers curl in the direction of the magnetic field, which is into the page, and our thumb points in the direction of the magnetic force, which is to the left. In other words, there is a magnetic force which opposes the motion of the conductor in the magnetic field. If the applied force is constant, the magnetic force will also be constant to keep the conductor moving at a constant velocity.
- Now we can use Faraday's law to determine the magnitude of the induced emf in the conductor.

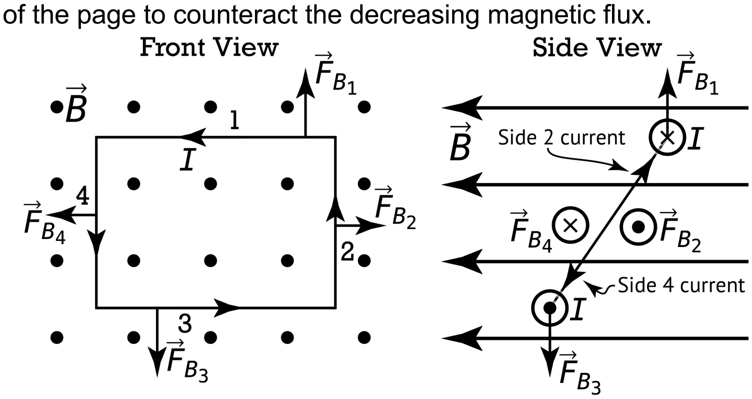
$$\varepsilon = -N\frac{d\Phi_B}{dt} = -N\frac{dBA\cos\theta}{dt} = -(1)B\cos(180°)\frac{d(Lx)}{dt} = BL\frac{dx}{dt}$$
$$\Rightarrow \varepsilon = vBL$$

Next, let's look at a conductive loop which has a current induced in it, something we talked about previously[1], that induced current is now a bunch of charge carriers which are moving in a magnetic field. Those moving charges now have induced forces acting on them, again this is something we talked about quite a before now[2]. The following equations determine that magnetic force:

$$\vec{F}_B = I\vec{L} \times \vec{B} \Rightarrow \|F_B\| = ILB\sin\theta$$

Let's walk through an example.
Below is a front view and a side view of a conducting loop in the shape of a rectangle. Let's start by only looking at the front view. Again, all directions for now refer to the *front view only*.

- A uniform magnetic field is directed out of the page and is decreasing.
- That means the magnetic flux through the loop is decreasing.
- Lenz' law states the induced B field is out of the page to counteract the decreasing magnetic flux.
- Using the alternate right-hand rule
  - Fingers curl with the induced B field, out of the page.
  - Thumb points counterclockwise with induced current.
- The right-hand rule on the induced current in side 1 of the loop:
  - Fingers point to the left in the direction of the induced current.
  - Fingers curl out of the page in the direction of the original magnetic field.
  - Thumb points up in the direction of the induced magnetic force on side 1.

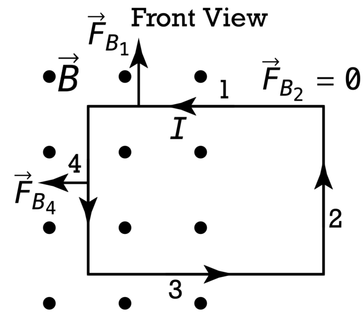



- For the remaining sides the right-hand rule shows the induced magnetic forces are:
  - To the right on side 2.
  - Down on side 3.
  - To the left on side 4.
- Notice that the net induced magnetic force on the loop equals zero!
  - The induced magnetic forces on sides 1 and 3 are equal and opposite.
  - The induced magnetic forces on sides 2 and 4 are equal and opposite.

[1] Electromagnetic Induction Review for AP Physics C: E&M- http://www.flippingphysics.com/apcem-electromagnetic-induction.html
[2] Magnetic Fields Review for AP Physics C: E&M - http://www.flippingphysics.com/apcem-magnetic-fields.html

Now let's switch to the side view. Again, all directions for now refer to the *side view only*.

- Side 1: The induced current is into the page, and the induced magnetic force is up.
- Side 2: The induced current is up and to the right, and the induced B force is out of the page.
- Side 3: The induced current is out of the page, and the induced magnetic force is down.
- Side 4: The induced current is down and to the left, and the induced B force is into the page.
- You can see the net induced magnetic force on the loop is still zero, however, …
- The induced magnetic forces cause a net torque on the loop! Net torque is not zero!
    - Assuming the loop is not attached to anything, the net torque on the loop would cause an angular acceleration around its center of mass which is counterclockwise at this specific moment in time.




Now let's change the example by making it so the magnetic field abruptly ends partway through the loop.[3]

- Again, the magnetic field is uniform, directed out of the page, and is decreasing in magnitude.
- Lenz' law gives us the same direction for the induced current in the loop; counterclockwise.
- Using the right-hand rule to determine the directions of the induced magnetic force:
    - Side 2: This entire side of the wire is not in the magnetic field, so there is *no induced magnetic force on side 2!*
    - Side 4: Everything is the same here. Induced magnetic force is to the left.
    - Sides 1 and 3: The directions are the same as before (1 is up, 3 is down), however, only the part of each side which is in the magnetic field will experience an induced magnetic force, therefore, the magnitudes of these induced magnetic forces are smaller than in the previous example.
- The net induced magnetic force on this loop *does not equal zero.*
    - The induced magnetic forces on sides 1 and 3 are equal and opposite.
    - The net force would accelerate the loop to the left.

In other words, the net induced magnetic force on a current carrying loop:

- Which is entirely in a uniform external magnetic field always equals zero.
    - (The induced magnetic forces can cause a net torque on the loop.)
- Which is only partially in a uniform magnetic field is nonzero.
    - This can cause a translational acceleration of the loop.

[3] I would argue that creating a magnetic field which looks like this is impossible, however, it is helpful for learning. So, step off!

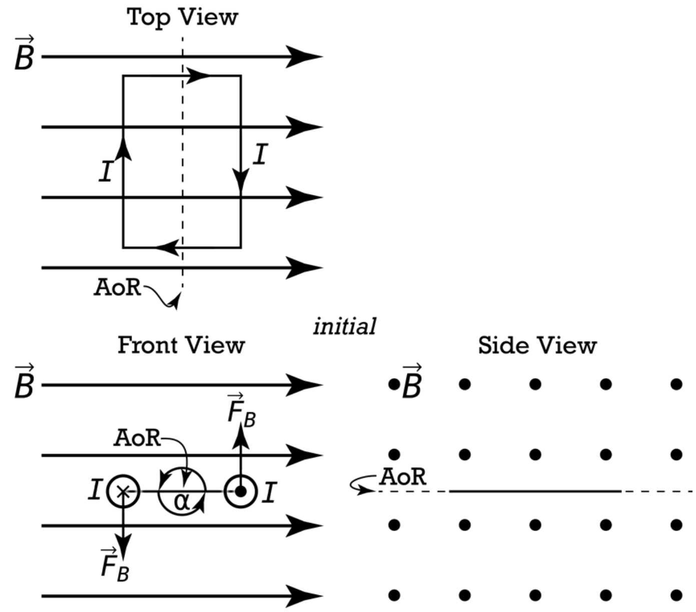
Let's switch it up again. In this example we have a rectangular conducting loop in a uniform magnetic field oriented as shown. We place an emf across the loop to cause current I in the loop.

In the *front view* you can see that, according to the right-hand rule, a net torque acts on the loop causing it to angularly accelerate in a clockwise direction (in the front view).

Everything we have been referring to is the initial position of the loop. After the loop has turned 90 degrees, we are now at the final position of the loop.

This is a very basic illustration of how an electric motor works. Current is placed through wire loops in magnetic fields which causes the loops to rotate converting electric potential energy to mechanical energy.

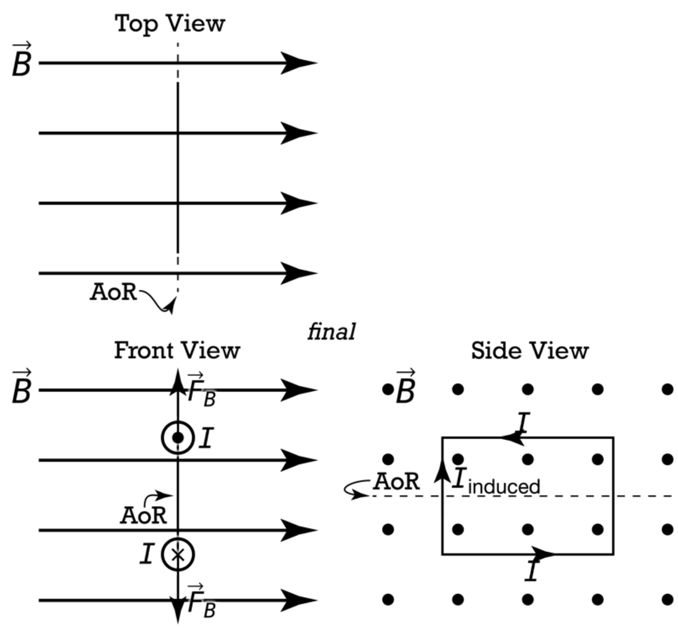
Now let's look at how the magnetic flux changes from the initial to final positions.
The initial magnetic flux through the loop is zero.
The final magnetic flux through the loop is nonzero.
The magnetic flux through the loop changes, which means there is an induced magnetic field, an induced emf, and an induced current in the loop. We need to use Lenz' law to determine the direction of the induced current.




In the side view, the magnetic flux is out of the page and increasing. In order to resist this change in magnetic flux, the induced magnetic field is into the screen (in the side view). According to the alternate right-hand rule, the fingers curl into the screen in the direction of the induced magnetic field inside the loop, thumb points clockwise (in the side view) in the direction of the induced current in the loop.

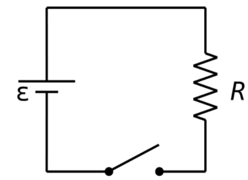
In other words, in electric motors, there is an induced emf and an induced current caused by the change in the magnetic flux in the loops of the motor, and that induced current is opposite the direction of the current placed in the loops to cause the loops to rotate. This induced current decreases the current in a turning electric motor. This concept is called *back emf* and is present in all electric motors when they are rotating.

Realize this back emf is not present when the electric motor is not rotating. In other words, when an electric motor is first starting up, the current through the electric motor is larger than when the electric motor is running at a constant angular velocity. This lack of back emf when an electric motor is not moving can cause lights which are on the same circuit to dim when an electric motor is first starting up and can even cause a circuit breaker to trip if something suddenly binds the electric motor causing it to stop rotating which brings the back emf to down zero and suddenly increases the current in the circuit above the maximum current allowed through the circuit breaker.[4]



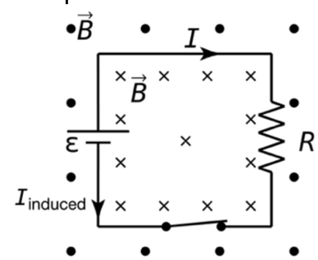

[4] Yes, I have done this. ☺️

Flipping Physics Lecture Notes:
Inductance
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-inductance.html




Let's look at a basic circuit. Before time t = 0, the switch in the circuit is open and zero current flows through the open loop. At time t = 0, the switch is closed and remains closed. From this perspective, a clockwise current, $I$, is now in the circuit. Up to this point we have assumed the current appears instantaneously in the circuit. You should realize that, in the real world, nothing changes instantaneously. So, let's look at what really happens when the switch closes.

According to the alternate right-hand rule, the clockwise current, $I$, in the circuit causes a magnetic field which is out of the page outside the loop and a magnetic field which is into the page inside the loop. In other words, this circuit is a loop which initially, before time t = 0, has zero magnetic flux in it and, as soon as the switch is closed, the loop has magnetic flux in it. We know, according to Faraday's law, that a changing magnetic flux induces an emf and can induce a current. We can use Lenz' law to determine the direction the induced current would be in the loop:




$$\varepsilon_{\text{induced}} = -N\frac{d\Phi_B}{dt}$$

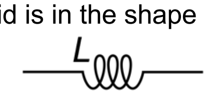
- Initially, there is zero magnetic flux.
- Finally, there is a B field which is into the page inside the loop.
  - Note: Only the magnetic field inside the loop causes a magnetic flux inside the loop.
- Therefore, the magnetic flux is increasing.
- Lenz's law states that an induced magnetic field is created to counteract the change in magnetic flux.
- Therefore, the induced magnetic field is out of the page.
- According to the alternate right-hand rule, an induced current would be counterclockwise in the loop from this perspective.
- This means the current in the circuit does not instantly change from 0 to $I$. The current in the circuit takes time to transition from 0 to $I$, because, the circuit itself opposes the change in current.
- This opposition of a circuit to a change in current in that same circuit is called *self-inductance*.
- In general, opposition to a change in current in a conductor is called *inductance.*

To get to the equation for inductance, we need to return to the simple circuit example and the basic concept of Faraday's law.

- Induced emf is proportional to change in magnetic flux with respect to time. $\varepsilon_{\text{induced}} \propto \frac{d\Phi_B}{dt}$
- The magnitude of magnetic flux equals the magnetic field times the area of the loop times the cosine of the angle between the direction of the magnetic field and the direction of the area. $\Phi_B = BA\cos\theta$
- Assuming the area and angle are not changing with respect to time, the induced emf is proportional to the change in the magnetic field with respect to time. $\varepsilon_{\text{induced}} \propto \frac{dB}{dt}$
- An example of a magnetic field around a current carrying wire is the one which surrounds an infinitely long current carrying wire which we have derived previously. $B = \frac{\mu_0 I}{2\pi a}$
  - "a" is the straight-line distance perpendicular out from the wire to the location of the B field.
- This means the induced emf in a conductor is proportional to change in current in the conductor with respect to time. $\varepsilon_{\text{induced}} \propto \frac{dI}{dt}$

An inductor is a circuit element with a known inductance.
The equation for the inductance of an inductor is: $\varepsilon_L = -L\frac{dI}{dt}$

- "L" is the inductance of the inductor.
- The simplest version of an inductor is a small, ideal solenoid. Because a solenoid is in the shape of a coil, the symbol for an inductor looks like the coils of a miniature solenoid.




- The units for inductance are henrys, H.
  - $\varepsilon_L = -L\frac{dI}{dt} \Rightarrow L = -\frac{\varepsilon_L}{dI/dt} \Rightarrow \frac{V}{A/s} \Rightarrow \text{henry}, H = \frac{V \cdot s}{A}$

It is important to understand the difference between resistance, resistivity, resistors, inductance, self-inductance, and inductors.

- *Resistance* is an opposition to current. *(concept)*

  $$R = \frac{\Delta V}{I}$$

  - The units for resistance are ohms, Ω.
  - The resistance of a circuit is often assumed to be zero. (self-resistance?)
  - A *resistor* is a circuit element with a specific resistance. *(physical object)*

    $$\rho = \frac{RA}{L}$$

    - "R" is the resistance of a resistor.
    - A resistor is made of a material with a material property called *resistivity*, ρ.
      - The units for resistivity are ohm meters, Ω•m.

      R

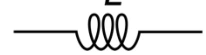
      

    - A resistor can be added to a circuit to change the resistance of the circuit.
    - A resistor can be added to a circuit diagram to model the resistance of the circuit itself.
- *Inductance* is an opposition to changes in current. *(concept)*
  - The units for inductance are henrys, H.
  - The opposition of a circuit to the change in the current in that circuit is called *self-inductance*. *(concept)*
    - The self-inductance of a circuit is often assumed to be zero.

    $$L = -\frac{\varepsilon_L}{\mathrm{d}I/\mathrm{d}t}$$

  - An *inductor* is a circuit element with a specific inductance. *(physical object)*
    - "L" is the inductance of an inductor.
    - A typical shape for an inductor is a small, ideal solenoid.

      L

      - There is no material property called "inductivity" because the inductance of an inductor is mostly caused by the shape, not the material, of the inductor. A magnetic material in its core can affect the inductance through its magnetic permeability, but not the material of the wire coil.
    - An inductor can be added to a circuit to change the inductance of the circuit.
    - An inductor can be added to a circuit diagram to model the self-inductance of the circuit itself.

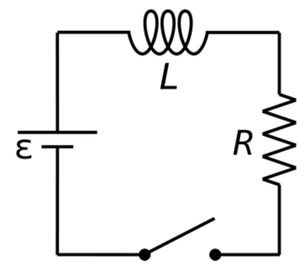
Considering the most common shape for an inductor is a small, ideal solenoid, let's look at that case specifically. We have two different equations for induced emf which we can set equal to one another:

- $$\varepsilon_{\text{induced}} = -N\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -L\frac{\mathrm{d}I}{\mathrm{d}t} \Rightarrow N\mathrm{d}\Phi_B = L\mathrm{d}I$$
  - N is the total number of loops or coils in the solenoid shaped inductor.
  - We can cancel out *dt* on both sides of the equation
- $$\Rightarrow \int N\mathrm{d}\Phi_B = \int L\mathrm{d}I \Rightarrow N\int_0^{\Phi_B}\mathrm{d}\Phi_B = L\int_0^{I}\mathrm{d}I \Rightarrow N\Phi_B = LI \Rightarrow L_{\text{solenoid}} = \frac{N\Phi_B}{I} = \frac{N(BA\cos\theta)}{I}$$
  - Take the integral of the whole equation.
  - Both N and L are constants and can be taken out from their integrals.
  - Substitute in the equation for the magnitude of magnetic flux.
- $$\Rightarrow L_{\text{solenoid}} = \frac{NBA\cos(0°)}{I} = B\left(\frac{NA}{I}\right) \;\;\&\;\; B_{\text{solenoid}} = \mu_0 nI = \frac{\mu_0 NI}{\ell}$$
  - In an ideal solenoid, angle between magnetic field and loop area vector is always 0°.
  - We have the equation for an ideal solenoid which we derived earlier.

    $$n = \frac{N}{\ell}$$

    - n is the turn density of the solenoid.
    - We already defined N as the total number of loops in the solenoid,
    - Therefore, the curly $\ell$, is the entire length of the ideal solenoid.
      - Note L ≠ $\ell$. (Inductance does not equal solenoid length.)
      - (L for a resistor is its length not its inductance. ☺)
- $$\Rightarrow L_{\text{solenoid}} = \left(\frac{\mu_0 NI}{\ell}\right)\left(\frac{NA}{I}\right) \Rightarrow L_{\text{solenoid}} = \frac{\mu_0 N^2 A}{\ell}$$
  - The inductance of an ideal solenoid is determined by:

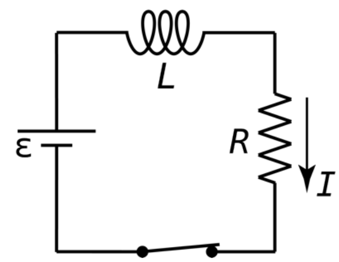
- N, the number of turns: A, the cross-sectional area: $\ell$, solenoid length.
- μ, the magnetic permeability of the space inside the solenoid. For an ideal solenoid with nothing inside it, that equals the magnetic permeability of free space.
- μ, the magnetic permeability of the core material, replaces $\mu_0$ when the solenoid has a core material such as iron.
  - Inductance does *not* depend on current through the solenoid!
    - Resistance does *not* depend on current either!




Next, we are going to derive the equation for the energy stored in the magnetic field generated in an inductor as charges move through the inductor. To do that, we need to discuss an LR circuit. A circuit with an inductor and a resistor in it. Initially, at time t < 0, the switch is open. At time t = 0, the switch is closed. The current will increase from zero to some steady-state current, I. We are not going to derive the time-dependent equations for LR circuits today, we will do that in a future lesson.

Using Kirchhoff's Loop Rule, starting from the lower left-hand corner we get:

- $$\Delta V = 0 = \varepsilon - \Delta V_L - \Delta V_R = \varepsilon - L\frac{dI}{dt} - IR \Rightarrow \varepsilon = L\frac{dI}{dt} + IR$$
  - Electric potential across the battery goes up because the battery is adding electric potential energy to the circuit.
  - Electric potential across the inductor goes down because electric potential energy is being stored in the magnetic field of the inductor.
  - Electric potential across the resistor goes down because the resistor dissipates electric potential energy from the system.
  - We can now multiply this whole equation by the circuit current, I.

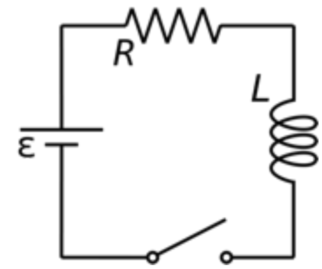



- $$\Rightarrow P = I\varepsilon = LI\frac{dI}{dt} + I^2R$$
  - We get the equation for power for each circuit element:
    - The rate at which energy is being added to the circuit by the battery.
    - The rate at which energy is being stored in the magnetic field of the inductor.
    - The rate at which energy is being dissipated by the resistor.
- We can now look specifically at the rate at which energy is being stored in the magnetic field of the inductor.
- $$\Rightarrow P = \frac{dU}{dt} = LI\frac{dI}{dt} \Rightarrow dU = LI\,(dI) \Rightarrow \int_0^{U_L} dU = \int_0^I LI\,(dI) = L\int_0^I I\,(dI)$$
  $$\Rightarrow U_L = \left[L\left(\frac{I^2}{2}\right)\right]_0^I \Rightarrow U_L = \frac{1}{2}LI^2$$
- We now have an equation for the energy stored in the magnetic field generated in an inductor as charges move through the inductor.
  - That energy is only present when current is passing through the inductor. This is because the magnetic field generated in the inductor is due to the charges moving through the inductor. If the charges are not moving, there is no magnetic field in the inductor.

A capacitor functions differently:

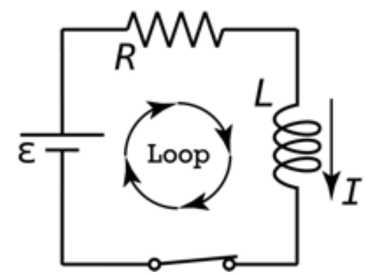
- The energy stored in a capacitor is stored in the electric field of the capacitor.
  - The energy stored in a capacitor can remain when a capacitor is disconnected from a circuit because charges can remain separated on the plates of the capacitor which would maintain the electric field between the plates of the capacitor.

$$U_C = \frac{Q^2}{2C} = \frac{1}{2}C\Delta V^2 = \frac{1}{2}Q\Delta V$$




This LR circuit is a circuit with a battery, a resistor, an inductor, and a switch. Before time t = 0, the switch is open. At time t = 0, the switch is closed and remains closed. A few general things to realize:

- The initial current in the circuit, at time t = 0, is zero.
- The inductor opposes the change in current in the circuit which is what causes the current to change from its initial current of zero to its final steady state current.
- After a long time, the inductor behaves like any other ideal wire in a circuit and has zero resistance. In other words, after a long time the current has reached its maximum value and behaves as if the inductor is not there.




Let's determine equations for the limits. To do so, we use Kirchhoff's Loop Rule starting in the lower left-hand corner of the circuit:

$$\Delta V_{\text{Loop}} = 0 = \varepsilon - \Delta V_R - \Delta V_L = \varepsilon - IR - L\frac{dI}{dt}$$

We can use this equation to determine the remaining limits.

$$@\ t_i = 0; I_i = 0$$

$$\Rightarrow 0 = \varepsilon - L\frac{dI}{dt} \Rightarrow \varepsilon = L\frac{dI}{dt} \Rightarrow \frac{dI}{dt} = \frac{\varepsilon}{L} \Rightarrow \left(\frac{dI}{dt}\right)_{\text{initial}} = \frac{\varepsilon}{L}\ \text{[max value]}$$

$$@\ t_f \approx \infty; \left(\frac{dI}{dt}\right)_{\text{final}} = 0 \Rightarrow 0 = \varepsilon - IR \Rightarrow I_f = \frac{\varepsilon}{R}\ \text{[max value]}$$

And now we can derive the equation for current in this LR circuit as a function of time. Going back to the Kirchhoff's Loop Rule equation:

$$0 = \varepsilon - IR - L\frac{dI}{dt} \Rightarrow L\frac{dI}{dt} = \varepsilon - IR \Rightarrow \frac{L}{R}\frac{dI}{dt} = \frac{\varepsilon}{R} - I$$

$$\&\ \text{Let } u = \frac{\varepsilon}{R} - I \Rightarrow du = -dI \Rightarrow \frac{L}{R}\frac{-du}{dt} = u \Rightarrow \frac{du}{u} = -\frac{R}{L}dt$$

$$\Rightarrow \int \frac{du}{u} = \int -\frac{R}{L}dt \Rightarrow \int_{u_i}^{u_f} \frac{1}{u}du = -\frac{R}{L}\int_0^t dt \Rightarrow \ln u\Big|_{u_i}^{u_f} = -\frac{R}{L}t\Big|_0^t$$

$$\&\ \int \frac{dx}{x-a} = \ln|x-a| \Rightarrow \int \frac{du}{u} = \ln|u| = \ln u$$

- In this problem a = 0 and, because I varies from 0 to $\frac{\varepsilon}{R}$, u is always positive (or zero).

$$\Rightarrow \ln u_f - \ln u_i = \ln\left(\frac{u_f}{u_i}\right) = -\frac{R}{L}t \Rightarrow e^{\left(\ln\left(\frac{u_f}{u_i}\right)\right)} = e^{\left(-\frac{R}{L}t\right)} \Rightarrow \frac{u_f}{u_i} = e^{\left(-\frac{Rt}{L}\right)}$$

$$\Rightarrow u_f = u_i e^{\left(-\frac{Rt}{L}\right)}\ \&\ u_f = \frac{\varepsilon}{R} - I_f\ \&\ u_i = \frac{\varepsilon}{R} - I_i = \frac{\varepsilon}{R}$$

$$\Rightarrow \frac{\varepsilon}{R} - I_f = \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)} \Rightarrow -I_f = -\frac{\varepsilon}{R} + \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)} \Rightarrow I_f = \frac{\varepsilon}{R} - \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)}$$

$$\Rightarrow I(t) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{Rt}{L}\right)}\right) = I_{\max}\left(1 - e^{\left(-\frac{Rt}{L}\right)}\right)$$

Note, this fits our limits because:

- $I(0) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{R(0)}{L}\right)}\right) = \frac{\varepsilon}{R}\left(1 - e^{0}\right) = \frac{\varepsilon}{R}(1-1) = 0$
- $I(\infty) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{R(\infty)}{L}\right)}\right) = \frac{\varepsilon}{R}\left(1 - e^{-\infty}\right) = \frac{\varepsilon}{R}(1-0) = \frac{\varepsilon}{R}$

We can also determine the time rate of change of current as a function of time:

$$I_f = \frac{\varepsilon}{R} - \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)} \Rightarrow \frac{dI}{dt} = \frac{d}{dt}\left(\frac{\varepsilon}{R} - \frac{\varepsilon}{R}e^{\left(-\frac{Rt}{L}\right)}\right) = \left(-\frac{\varepsilon}{R}\right)\frac{d}{dt}e^{\left(-\frac{Rt}{L}\right)} = -\left(\frac{\varepsilon}{R}\right)\left(\frac{R}{L}\right)e^{\left(-\frac{Rt}{L}\right)}$$

$$\Rightarrow \frac{dI}{dt}(t) = \frac{\varepsilon}{L}e^{\left(-\frac{Rt}{L}\right)} = \left(\frac{dI}{dt}\right)_{\max} e^{\left(-\frac{Rt}{L}\right)} \quad \& \quad \frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$

Again, this fits our limits because:

$$\frac{dI}{dt}(0) = \frac{\varepsilon}{L}e^{\left(-\frac{R(0)}{L}\right)} = \frac{\varepsilon}{L}e^{0} = \frac{\varepsilon}{L} \quad \& \quad \frac{dI}{dt}(\infty) = \frac{\varepsilon}{L}e^{\left(-\frac{R(\infty)}{L}\right)} = \frac{\varepsilon}{L}e^{-\infty} = 0$$

And we can determine the time constant, $\tau$:

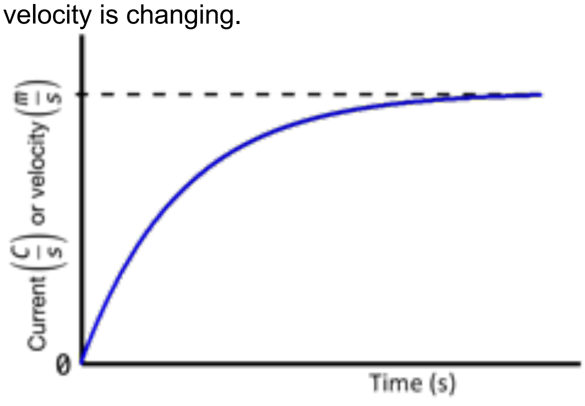
$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{t}{\tau}\right)}\right) \quad \& \quad \frac{dI}{dt}(t) = \frac{\varepsilon}{L}e^{\left(-\frac{t}{\tau}\right)} \Rightarrow \tau = \frac{L}{R}$$

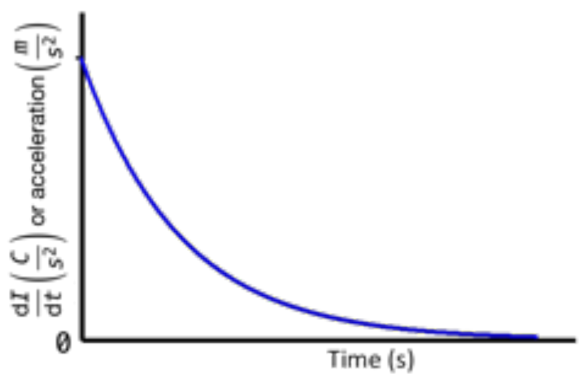
$$\Rightarrow I(\tau) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{\tau}{\tau}\right)}\right) = \frac{\varepsilon}{R}\left(1 - e^{-1}\right) = \frac{\varepsilon}{R}(1 - 0.368) = 0.632\frac{\varepsilon}{R}$$

$$\Rightarrow \frac{dI}{dt}(\tau) = \frac{\varepsilon}{L}e^{\left(-\frac{\tau}{\tau}\right)} = \frac{\varepsilon}{L}e^{(-1)} = 0.368\frac{\varepsilon}{L} = (1 - 0.632)\frac{\varepsilon}{L}$$

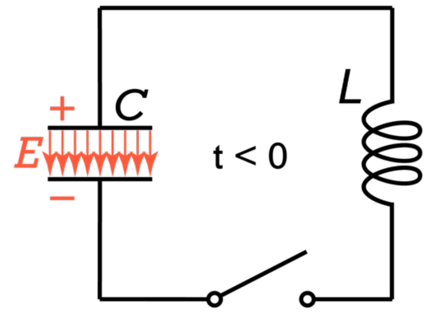
And the graphs:

You can consider the derivative of current with respect to time to be the acceleration of moving objects. Amps are coulombs per second. So, amps per second are coulombs per second squared. The time rate of change of current is the rate at which the current is changing, just like acceleration is the rate at which velocity is changing.



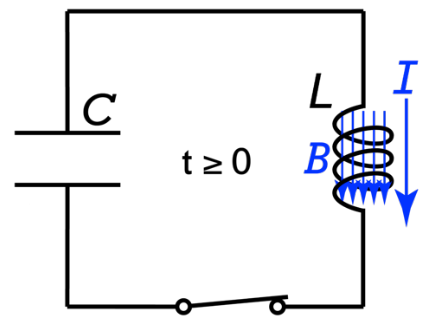



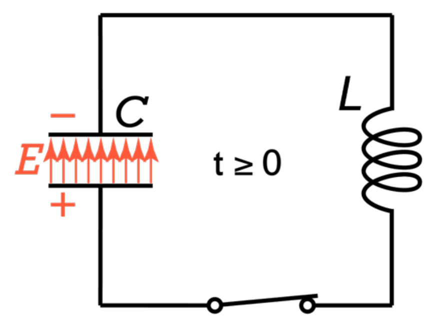
Flipping Physics Lecture Notes:
LC Circuits
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-lc-circuits.html




This LC circuit is a circuit with a capacitor, an inductor, and a switch. Before time t = 0, the switch was open for a long time. At time t = 0, the switch is closed and remains closed. A few general things to realize:

- The initial charge on the capacitor must be nonzero, if it were zero, nothing would happen when the switch is closed.
- The initial current in the circuit must be zero because there was no current in the open circuit before the switch was closed.
- The inductor opposes the change in current in the circuit which is why it takes time for the current to change from zero.
- The current through the inductor is from the charges leaving the capacitor to flow through the circuit, therefore, as current through the inductor increases, charge on the capacitor decreases.
- The electric field in the capacitor is decreasing in magnitude and the magnetic field in the inductor is increasing in magnitude.
- Once the charge is completely discharged, q = 0, the inductor has its maximum magnitude magnetic field and the current through the inductor is at its maximum.
- Current will continue to flow and build up charges on the plates of the capacitor, however, the orientation of the positive and negative plates will be reversed, and the current is decreasing.
- Eventually the current through the inductor will reduce to zero and charge will be at a maximum on the plates of the capacitor.
- Repeat the whole cycle in reverse.
- This is *simple harmonic motion!*
    - A horizontal mass-spring system is a good analogous situation.




Now let's derive equations for the LC Circuit, starting with the total energy in the circuit:

$$U_t = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{q^2}{2C} + \frac{1}{2}Li^2$$

- Typically, we use uppercase symbols for constants and lowercase symbols for variables.
- We know $I_{\text{max}} \rightarrow q = 0 \;\&\; Q_{\text{max}} \rightarrow i = 0$
- We can take the derivative with respect to time of the total energy equation. We know the derivative of total energy in the LC circuit equals zero because these are all ideal components with zero resistance. In other words, no energy is being dissipated from the system.

$$\Rightarrow \frac{dU_t}{dt} = \frac{d}{dt}\left(\frac{q^2}{2C} + \frac{1}{2}Li^2\right) = 0$$

- We need to use the chain rule for both energy expressions because time is not a variable in either energy expression, however, both charge, q, and current, i, are changing with respect to time.

$$\Rightarrow 0 = \frac{d}{dt}\left(\frac{q^2}{2C}\right) + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) \Rightarrow 0 = \left(\frac{2q}{2C}\right)\frac{dq}{dt} + \left(\frac{2Li}{2}\right)\frac{di}{dt}$$

$$\Rightarrow 0 = \left(\frac{q}{C}\right)\frac{dq}{dt} + (Li)\frac{di}{dt} \;\;\&\;\; i = \frac{dq}{dt} \;\;\&\;\; \frac{di}{dt} = \frac{d^2q}{dt^2}$$

$$\Rightarrow 0 = \left(\frac{q}{C}\right)i + (Li)\frac{d^2q}{dt^2} = \frac{q}{C} + (L)\frac{d^2q}{dt^2} \Rightarrow -\frac{q}{C} = (L)\frac{d^2q}{dt^2} \Rightarrow \frac{d^2q}{dt^2} = -\frac{1}{LC}q$$

- The equation definition for simple harmonic motion is: $\frac{d^2x}{dt^2} = -\omega^2 x$
- Therefore, we know the angular frequency of an LC circuit And we can determine the period of an LC Circuit:

$$\Rightarrow \omega_{LC}^2 = \frac{1}{LC} \Rightarrow \omega_{LC} = \frac{1}{\sqrt{LC}}$$

$$\&\;\; \omega = 2\pi f = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} \Rightarrow T_{LC} = \frac{2\pi}{1/\sqrt{LC}} \Rightarrow T_{LC} = 2\pi\sqrt{LC}$$

- And we know a general equation which satisfies the simple harmonic motion equation:

$$x(t) = A\cos(\omega t + \phi) \Rightarrow q(t) = Q_{\max}\cos\left(\frac{t}{\sqrt{LC}} + \phi\right) \Rightarrow q(t) = Q_{\max}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

- For this specific LC circuit the initial charge on the capacitor is $Q_{max}$ , therefore, the phase constant is zero.

$$\&\;\; i = \frac{dq}{dt} \Rightarrow i(t) = \frac{d}{dt}\left[Q_{\max}\cos\left(\frac{t}{\sqrt{LC}}\right)\right] = -Q_{\max}\sin\left(\frac{t}{\sqrt{LC}}\right)\frac{d}{dt}\left(\frac{t}{\sqrt{LC}}\right)$$

- We can also determine current in an LC circuit as a function of time and an equation relating current maximum to charge maximum.

$$\Rightarrow i(t) = -\frac{Q_{\max}}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) \Rightarrow I_{\max} = \frac{Q_{\max}}{\sqrt{LC}} \Rightarrow i(t) = -I_{\max}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

- We can also derive the current maximum using the equation for total energy in the LC circuit.

$$U_t = U_C + U_L = \frac{q^2}{2C} + \frac{1}{2}Li^2 = \frac{Q_{\max}^2}{2C} + 0 = 0 + \frac{1}{2}LI_{\max}^2 \Rightarrow \frac{Q_{\max}^2}{C} = LI_{\max}^2$$

$$\Rightarrow I_{\max}^2 = \frac{Q_{\max}^2}{LC} \Rightarrow I_{\max} = \frac{Q_{\max}}{\sqrt{LC}}$$

- We can determine equations for energy as functions of time.

$$U_C = \frac{q^2}{2C} \Rightarrow U_C(t) = \frac{\left[Q_{\max}\cos\left(\frac{t}{\sqrt{LC}}\right)\right]^2}{2C} \Rightarrow U_C(t) = \frac{Q_{\max}^2}{2C}\cos^2\left(\frac{t}{\sqrt{LC}}\right)$$

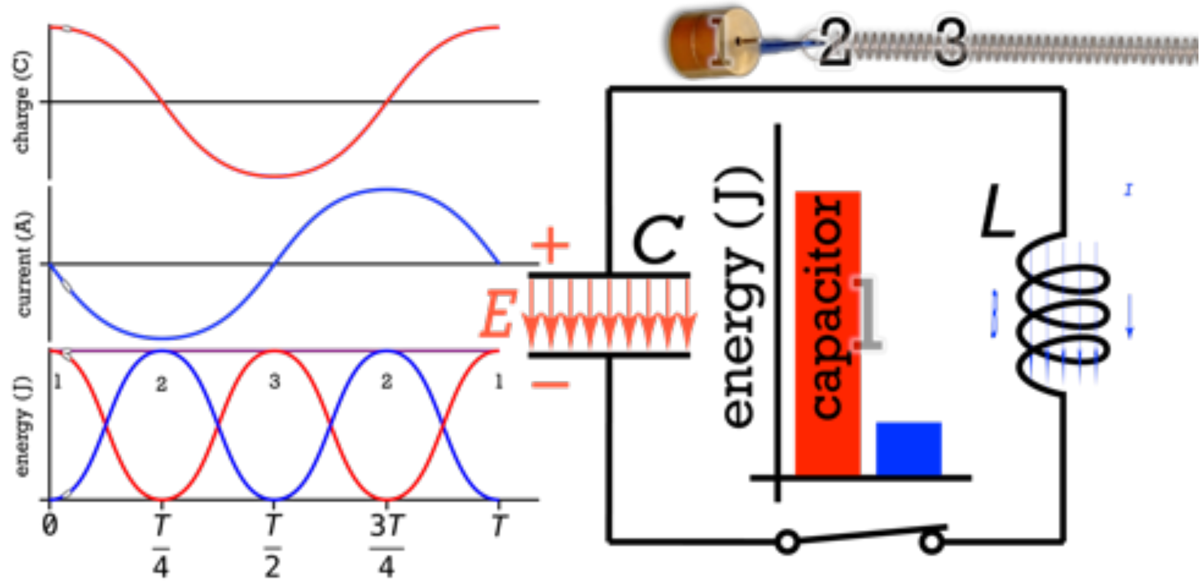
$$U_L = \frac{1}{2}Li^2 \Rightarrow U_L(t) = \frac{1}{2}L\left[-\frac{Q_{\max}}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)\right]^2 = \left(\frac{1}{2}L\right)\left(\frac{Q_{\max}^2}{LC}\right)\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

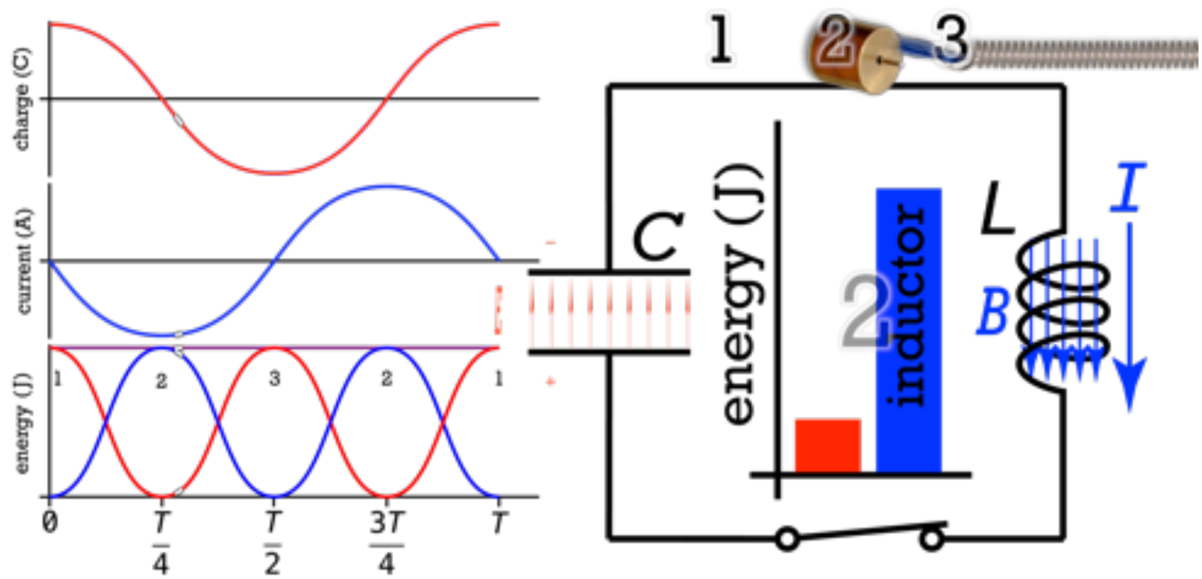
$$\Rightarrow U_L(t) = \frac{Q_{\max}^2}{2C}\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$U_t(t) = U_C(t) + U_L(t) = \frac{Q_{max}^2}{2C}\cos^2\left(\frac{t}{\sqrt{LC}}\right) + \frac{Q_{max}^2}{2C}\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$\Rightarrow U_t(t) = \left(\frac{Q_{max}^2}{2C}\right)\left[\cos^2\left(\frac{t}{\sqrt{LC}}\right) + \sin^2\left(\frac{t}{\sqrt{LC}}\right)\right] \Rightarrow U_t(t) = \frac{Q_{max}^2}{2C} \quad \& \quad \sin^2\theta + \cos^2\theta = 1$$

Below are two screenshots of the LC circuit animation. Honestly, you need to watch and hear the discussion of everything going on the animation to understand it.



Flipping Physics Lecture Notes:
Equations to Memorize for
AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-equations.html

I am definitely not a fan of rote memorization, however, sometimes there are good reasons to memorize equations. Here are my suggestions for AP Physics C: Electricity and Magnetism memorization

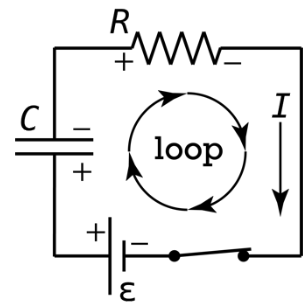
- Quantization of charge: $Q = ne$
  - e = elementary charge: $e = 1.60 \times 10^{-19} C$
- The Law of Charges:
  - Two charges with opposite signs attract one another.
  - Two charges with the same sign repel one another.
- The electric field around a point charge:
  - $$\vec{E} = \frac{\vec{F}_e}{q} \ \& \ F_e = \frac{kq_1q_2}{r^2} \Rightarrow E_{\text{point charge}} = \frac{\frac{kqQ}{r^2}}{q} = \frac{kQ}{r^2}$$
- You do not have to memorize it, it is on the Table of Information, however, it seems to come up quite often. The relationship between the Coulomb's law constant and vacuum permittivity:
  - $$k = \frac{1}{4\pi\varepsilon_0}$$
- Electric field around a continuous charge distribution: (memorize and know how to derive)
  - $$\vec{E}_{\text{point charge}} = \frac{kQ}{r^2}\hat{r} \Rightarrow d\vec{E} = \frac{k\,(dq)}{r^2}\hat{r} \Rightarrow \int d\vec{E} = \int \frac{k\,(dq)}{r^2}\hat{r}$$
  - $$\Rightarrow \vec{E}_{\text{continuous charge distribution}} = k\int \frac{dq}{r^2}\hat{r}$$
- The charge densities:
  - linear charge density, $\lambda = \frac{Q}{L} = \frac{dq}{dL}$ in $\frac{C}{m}$
  - surface charge density, $\sigma = \frac{Q}{A} = \frac{dq}{dA}$ in $\frac{C}{m^2}$
  - volumetric charge density, $\rho = \frac{Q}{V} = \frac{dq}{dV}$ in $\frac{C}{m^3}$
- The symbol for electric flux of Gauss' law never appears on the equation sheet and it does not clarify that the charge is the charge enclosed in the Gaussian surface:
  - $$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$
- If the net charge inside a closed Gaussian surface is zero, then the net electric flux through the Gaussian surface is zero. $$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0} = \frac{0}{\varepsilon_0} = 0$$
- Equation for electric flux for a flat surface in a uniform electric field is not on the equation sheet:
  - $$\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta$$
- The electric potential difference across a uniform electric field. Remember d is the straight-line distance parallel to the electric field.
  - $$\Delta V = -\int \vec{E} \cdot d\vec{r} = -\int_a^b E\cos(0^\circ)\,dr = -E\int_a^b dr \Rightarrow \Delta V_{\text{uniform } E} = -Ed$$
- They only have 2 out of the 3 energy stored in a capacitor equations:

- $$U_C = \frac{1}{2}Q\Delta V = \frac{1}{2}C\Delta V^2 = \frac{Q^2}{2C}$$

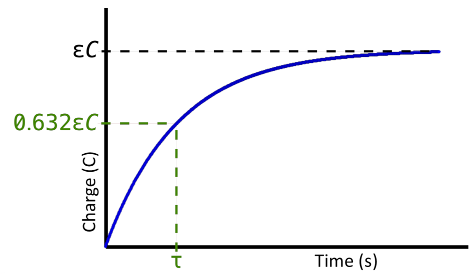
- Only 1 of the equations for electric power is on the equation sheet:

$$P = I\Delta V = I^2 R = \frac{\Delta V^2}{R}$$

- When we add a resistor in series, the equivalent resistance increases.
- When we add a resistor in parallel, the equivalent resistance decreases.
  - $$R_{\text{eq series}} = \sum_n R_n \quad \& \quad \frac{1}{R_{\text{eq parallel}}} = \sum_n \frac{1}{R_n}$$
- When we add a capacitor in series, the equivalent capacitance decreases.
- When we add a capacitor in parallel, the equivalent capacitance increases.
  - $$\frac{1}{C_{\text{eq series}}} = \sum_n \frac{1}{C_n} \quad \& \quad C_{\text{eq parallel}} = \sum_n C_n$$
  - We can see the relationships from the equations provided on the equation sheet, however, it comes up often enough that it is good to memorize and, if you need to confirm what you memorized, just look at the equation sheet.

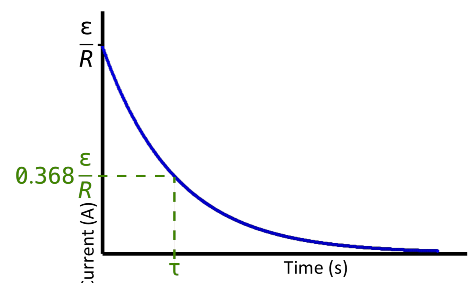



- None of the equations for RC or LR circuits are on the equation sheet, however, I do not suggest you memorize them. Instead, I suggest you know how to find their limits, know their general shapes, memorize the time constants, and memorize that one time constant represents the time for a 63.2% change.



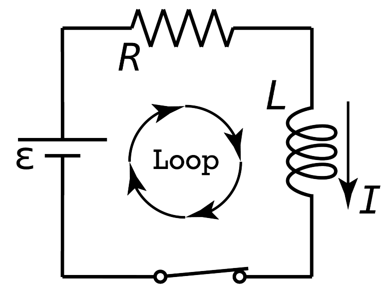

  - $$\Delta V_{\text{loop}} = 0 = \varepsilon - \Delta V_C - \Delta V_R \Rightarrow \varepsilon = \frac{Q}{C} + IR$$
  - $$\Rightarrow Q_i = 0 \rightarrow I_i = \frac{\varepsilon}{R}$$

$$\& \ I_f = 0 \rightarrow Q_f = \varepsilon C$$




  - $$y = (1 - e^{-x}) \Rightarrow q(t) = \varepsilon C\left(1 - e^{-\frac{t}{RC}}\right)$$

- $y = e^{-x} \Rightarrow i(t) = \frac{dq}{dt} = \frac{\varepsilon}{R} e^{-\frac{t}{RC}}$
- $\tau_{RC} = RC$
- $y = 1 - e^{-x} \Rightarrow 1 - e^{-1} = 0.632$
  - One time constant represents the time for a 63.2% change.




- We can do the same thing for an LR circuit:
  - $\Delta V_{\text{loop}} = 0 = \varepsilon - \Delta V_R - \Delta V_L \Rightarrow \varepsilon = IR + L\frac{dI}{dt}$
  - $\Rightarrow I_i = 0 \to \left(\frac{dI}{dt}\right)_{\text{initial}} = \frac{\varepsilon}{L}$ & $\left(\frac{dI}{dt}\right)_{\text{final}} = 0 \to I_f = \frac{\varepsilon}{R}$ & $\tau_{LR} = \frac{L}{R}$
  - $y = (1 - e^{-x}) \Rightarrow I(t) = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{Rt}{L}\right)}\right)$ & $y = e^{-x} \Rightarrow \frac{dI}{dt}(t) = \frac{\varepsilon}{L} e^{\left(-\frac{Rt}{L}\right)}$

- For Ampère's law, the equation sheet does not identify it is the current inside the Amperian loop:
  - $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{in}}$
- Motional emf. But of course, you need to know how to derive this motional emf equation and remember this assumes the velocity and magnetic field are at right angles relative to one another.
  - $\varepsilon = vBL$
    - In this motional emf equation, "L" stands for the length of the conductor.
- The inductance of an ideal solenoid (and how to derive it). Mostly so you remember what inductance does and does not depend on:
  - $L_{\text{solenoid}} = \frac{\mu_0 N^2 A}{\ell}$
  - The inductance of an ideal solenoid is determined by:
    - N, the number of turns: A, the cross-sectional area: $\ell$, solenoid length.
    - μ, the magnetic permeability of the space inside the solenoid. For an ideal solenoid with nothing inside it, that equals the magnetic permeability of free space.
    - μ, the magnetic permeability of the core material, replaces $\mu_0$ when the solenoid has a core material such as iron.
      - Inductance does *not* depend on current through the solenoid!
        - Resistance does *not* depend on current.
        - Capacitance does *not* depend on charge on the plates or ΔV across the plates.
- The angular frequency of LC circuits, and therefore, all the simple harmonic motion equations for LC circuits.
  - Energy oscillates back and forth between electric potential energy stored in the electric field of the capacitor and magnetic potential energy stored in the magnetic field of the inductor.
  - $\omega_{LC} = \frac{1}{\sqrt{LC}} = 2\pi f = \frac{2\pi}{T} \Rightarrow T_{LC} = 2\pi\sqrt{LC}$
  - $q(t) = Q_{\max}\cos(\omega t + \phi) \Rightarrow i(t) = \frac{dq}{dt} = -I_{\max}\sin(\omega t + \phi)$
    - (The above Simple Harmonic Motion equations did not make the video. It's *probably* not worth memorizing them.)