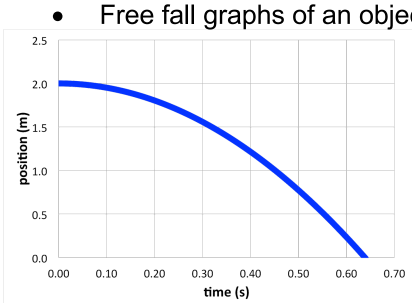
Flipping Physics Lecture Notes:
AP Physics C: Kinematics Review (Mechanics)
https://www.flippingphysics.com/apc-kinematics-review.html

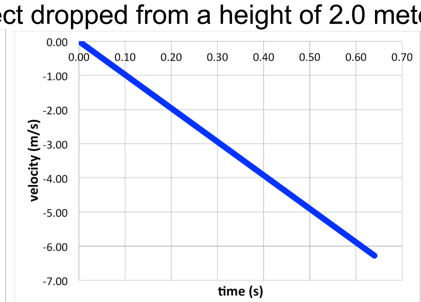
- Dimensions are your friends!!:
    - Be careful with your conversions and give all your answers units (when they have them!).
    - $\rho_{Krypton} = 3.75\dfrac{g}{cm^3}\left(\dfrac{100cm}{1m}\right)^3\left(\dfrac{1kg}{1000g}\right) = \boxed{0.375\dfrac{kg}{m^3}}$
- Vector vs. Scalar:
    - Vectors have both magnitude and direction.
    - Scalars have magnitude only (no direction) but can be positive or negative.
- Instantaneous velocity is the derivative of position as a function of time: $\vec{v}_{instantaneous} = \dfrac{d\vec{x}}{dt}$
    - Not to be confused with average velocity: $\vec{v}_{average} = \dfrac{\Delta\vec{x}}{\Delta t}$
- Instantaneous acceleration is the derivative of velocity as a function of time: $\vec{a}_{instantaneous} = \dfrac{d\vec{v}}{dt}$
    - Not to be confused with average acceleration: $\vec{a}_{average} = \dfrac{\Delta\vec{v}}{\Delta t}$
- The derivative represents the slope of the line.
- Uniformly Accelerated Motion or UAM.

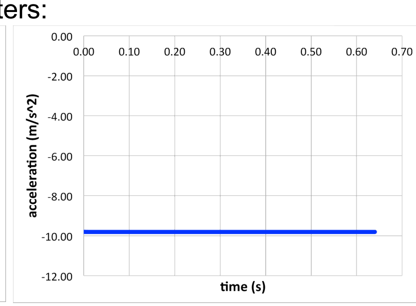
AP® Physics C Equation Sheet	*Flipping Physics®*
$v_x = v_{x0} + a_x t$	$v_f = v_i + a\Delta t$
$x = x_0 + v_{xo}t + \frac{1}{2}a_x t^2$	$\Delta x = v_i\Delta t + \frac{1}{2}a\Delta t^2$
$v_x^{\ 2} = v_{x0}^{\ 2} + 2a_x\left(x - x_0\right)$	$v_f^{\ 2} = v_i^{\ 2} + 2a\Delta x$
	$\Delta x = \frac{1}{2}\left(v_f + v_i\right)\Delta t$

- The AP Physics C UAM Equations assume $t_i = 0$; $\Delta t = t_f - t_i = t_f - 0 = t$
- x_0 means the initial position.

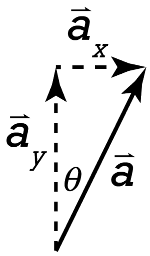
- Free fall is when the only force acting on an object is the Force of Gravity. (No air resistance)
    - $a_y = -g = -9.81\dfrac{m}{s^2}$ & $g_{Earth} = +9.81\dfrac{m}{s^2}$
    - This is Uniformly Accelerated Motion where you already know the acceleration!
- Free fall graphs of an object dropped from a height of 2.0 meters:







- Component vectors are the vectors in the x, y (and possibly z) directions that make up a vector.



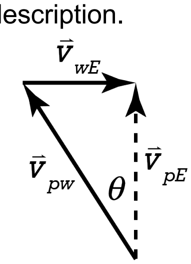
  - $\vec{a} = 2.5\frac{m}{s^2}@26^\circ\, E\ of\ N$
    - $\sin\theta = \frac{O}{H} = \frac{a_x}{a} \Rightarrow a_x = a\sin\theta = 2.5\sin(26) = 1.09593 \approx \boxed{1.1\frac{m}{s^2}}$
    - $\cos\theta = \frac{A}{H} = \frac{a_y}{a} \Rightarrow a_y = a\cos\theta = 2.5\cos(26) = 2.24699 \approx \boxed{2.2\frac{m}{s^2}}$
- Unit Vectors $\hat{i}, \hat{j}, and\ \hat{k}$ are vectors with a value of 1 in the x, y, and z directions respectively.
  - In other words the acceleration in the above example in unit vector form is:

    $\vec{a} = \left[1.1\hat{i} + 2.2\hat{j}\right]\frac{m}{s^2}$ This is the same as $\vec{a} = 2.5\frac{m}{s^2}@26^\circ\, E\ of\ N$
- Vector addition is much easier using Unit Vectors. Example:

  $\vec{A} = \left[1.00\hat{i} + 2.00\hat{j}\right]m;\ \vec{B} = \left[2.50\hat{i} - 1.50\hat{j}\right]m;\ \vec{C} = \left[3.00\hat{i} + 3.50\hat{j}\right]m$

  $\vec{R} = \vec{A} + \vec{B} + \vec{C} = ? = \left[1.00\hat{i} + 2.00\hat{j}\right] + \left[2.50\hat{i} - 1.50\hat{j}\right] + \left[3.00\hat{i} + 3.50\hat{j}\right]$

  $\Rightarrow \vec{R} = \left[1 + 2.5 + 3\right]\hat{i} + \left[2 - 1.5 + 3.5\right]\hat{j} = \left[6.50\hat{i} + 4.00\hat{j}\right]m$
- $\vec{r}$ is generally used as the position vector symbol. This is so you can give the position of an object in 2 (or even 3) dimensions. For example if an object is located -2.0 meters in the x-direction, 7.4 meters in the y-direction and -3.7 meters in the z-direction, we can illustrate its position as $\vec{r} = \left[-2.0\hat{i} + 7.4\hat{j} + -3.7\hat{k}\right]m$.
  - Note: The position vector equation $\vec{r}$ is much shorter than the word description.
- Relative velocity is simply vector addition. $\vec{v}_{pE} = \vec{v}_{pw} + \vec{v}_{wE}$



  - The velocity of the plane with respect to the Earth equals the velocity of the plane with respect to the wind plus velocity of the wind with respect to the Earth.
  - Don't forget it's tip-to-tail vector addition. So $\vec{v}_{pw} + \vec{v}_{wE}$ are drawn "tip-to-tail".
- Projectile Motion is when the only force acting on an object is the Force of Gravity and the object is moving in both the x and y directions. (No air resistance)

x direction	*y direction*
$a_x = 0$	Free-Fall
Constant Velocity	$a_y = -g = -9.81\frac{m}{s^2}$
$v_x = \frac{\Delta x}{\Delta t}$	Uniformly Accelerated Motion
Δt (or t) is the same in both directions because it is a *scalar* and has magnitude only (no direction).	

  - Break the initial velocity into its components.

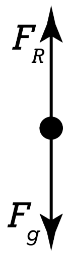
Flipping Physics Lecture Notes:

AP Physics C: Dynamics Review (Mechanics)
https://www.flippingphysics.com/apc-dynamics-review.html

- Newton's 1st Law: When viewed from an inertial reference frame, an object at rest will remain at rest and an object in motion will remain at a constant velocity unless acted upon by a net external force.
  - An inertial reference frame is where the acceleration of the reference frame zero.
  - A non-inertial reference frame is where the acceleration of the reference frame is not zero.
  - Also called the "Law of Inertia".
    - Inertia is the tendency of an object to resist acceleration.
- Newton's 2nd Law: $\sum \vec{F} = m\vec{a}$ on the equation sheet it is $\vec{a} = \frac{\sum \vec{F}}{m} = \frac{\vec{F}_{net}}{m}$.
- Newton's 3rd Law: $\vec{F}_{12} = -\vec{F}_{21}$
- $\sum \vec{F} = m\vec{a} \Rightarrow newtons,\ N = \frac{kg \cdot m}{s^2}$
- The basic forces with which we begin dynamics:
  - Force of Gravity also called Weight. $F_g = mg$
    - The force of gravity is caused by the interaction between the object and the planet.
    - The force of gravity is always down.
    - The acceleration due to gravity, g, her on planet Earth is +9.81 m/s^2.
    - Sometimes the symbol is W.
    - The force of gravity acts on the center of gravity of the object. (Which is the same as the center of mass in a constant gravitational field like the one we live in.)
  - Force Normal, F_N: A pushing force caused by a surface.
    - The force normal is normal to (perpendicular to) the surface.
    - The force normal is always a push. (Never a pull. A surface can't "pull".)
    - The force normal acts on the contact point between the two surfaces.
  - Force of Tension, F_T: The force caused by a rope, cable, wire, string, etc.
    - Always in the direction of the rope, cable, wire, string, etc.
    - Always a pull. (Never a push. You can't "push" with a rope.)
    - Sometimes the symbol is T.
  - Force Applied, F_a: The force of one object pushing or pulling on another object.
  - Force of Friction, F_f: The force caused by the interaction between two surfaces.
    - With regards to the direction of the Force of Friction. F_f always:
      - is parallel to the surface.
      - opposes motion (opposes sliding between the two surfaces)
      - is independent of the direction of the Force Applied.
    - General formula on the equation sheet: $\left|\vec{F}_f\right| \le \mu \left|\vec{F}_N\right|$
    - Static friction is when the two surfaces do NOT slide relative to one another.
      - $\vec{F}_{sf} \le \mu_s \vec{F}_N$ & $\vec{F}_{sf_{\max}} = \mu_s \vec{F}_N$
    - Kinetic friction is when the two surfaces DO slide relative to one another.
      - $\vec{F}_{kf} = \mu_k \vec{F}_N$

        - The coefficient of friction, μ, is an experimentally determined, dimensionless number which depends on the materials of the two interacting surfaces.
            - General range is 0 – 2:
                - However, μ can get up to 4 in extreme circumstances.
            - $\mu_s > \mu_k$ (For the same two interacting materials.)
- Free Body Diagrams or Force Diagrams. The five steps are …
    1. Draw the Free Body Diagram(s).
    2. Break forces in to components.
    3. Redraw the Free Body Diagram(s).
    4. Sum the forces.
    5. Sum the forces (in a direction perpendicular to the direction in step 4).
        - Only forces are drawn in Free Body Diagrams.
        - When on an incline we will often break the force of gravity in to it's parallel and perpendicular components and sum the forces in the parallel and perpendicular directions. $F_{g_\perp} = mg\cos\theta$ & $F_{g_\parallel} = mg\sin\theta$
        - Always draw the Free Body Diagram without breaking forces into components first and then redraw the Free Body Diagram. These are specific instructions from The AP CollegeBoard!
        - When summing the forces you must identify:
            - Positive directions, especially for pulleys!
            - Which object(s) you are summing the forces on.
            - Which direction you are summing the forces in.
        - You can only sum the forces on multiple objects at the same time if they all have the same acceleration.
- Translational equilibrium.
    - Translational motion simply means moving from one location to another.
    - Translational Equilibrium means the net force acting on the object is zero, $\sum \vec{F} = 0$.
    - An object in translational equilibrium is not accelerating.
        - $\sum \vec{F} = 0 = m\vec{a} \Rightarrow \vec{a} = 0$.
            - The object moves with a constant velocity or is at rest.
- The Drag Force or the Resistive Force, F_R: The force caused by the interaction of an object and the fluid the object is moving through.
    - Sometimes the symbol is R or F_D.
    - Opposite the direction of motion of the object.
    - For "small" objects moving at "slow" speeds, $\vec{F}_R = -b\vec{v}$.
        - The resistive force equals the negative of, b, the proportionality constant times the velocity of the object.
    - For all other objects (and more generally applicable), $\vec{F}_R = \frac{1}{2}D\rho A v^2$.
        - D is the Drag Coefficient of the object, has no dimensions, is experimentally determined, and depends on the shape and surface texture of the object.
        - ρ is the density *of the medium* through which the object is moving.
        - A is the cross sectional area of the object normal to the direction of motion.
        - v is the velocity of the object.

- Terminal velocity is when an object moving through a fluid has reached translational equilibrium. Force example an object which is falling downward in the Earth's atmosphere has a free body diagram with the force of gravity down and the resistive force up.



  - $\sum F_y = F_R - F_g = ma_y \Rightarrow \frac{1}{2}D\rho Av^2 - mg = ma_y \Rightarrow a_y = \frac{\frac{1}{2}D\rho Av^2 - mg}{m} = \frac{D\rho Av^2}{2m} - g$
    - In other words, in the absence of air resistance, $a_y = -g$!!!
  - With $a_y = 0$ $\Rightarrow 0 = \frac{D\rho Av^2}{2m} - g \Rightarrow \frac{D\rho Av^2}{2m} = g \Rightarrow \boxed{v_{terminal} = \sqrt{\frac{2mg}{D\rho A}}}$
  - Note: This equation is only true for "an object which is falling downward in the Earth's atmosphere". A rocket moving upward will have a different equation for terminal velocity because the free body diagram is different.

Flipping Physics Lecture Notes:

AP Physics C: Work, Energy, and Power Review (Mechanics)
https://www.flippingphysics.com/apc-work-energy-power-review.html

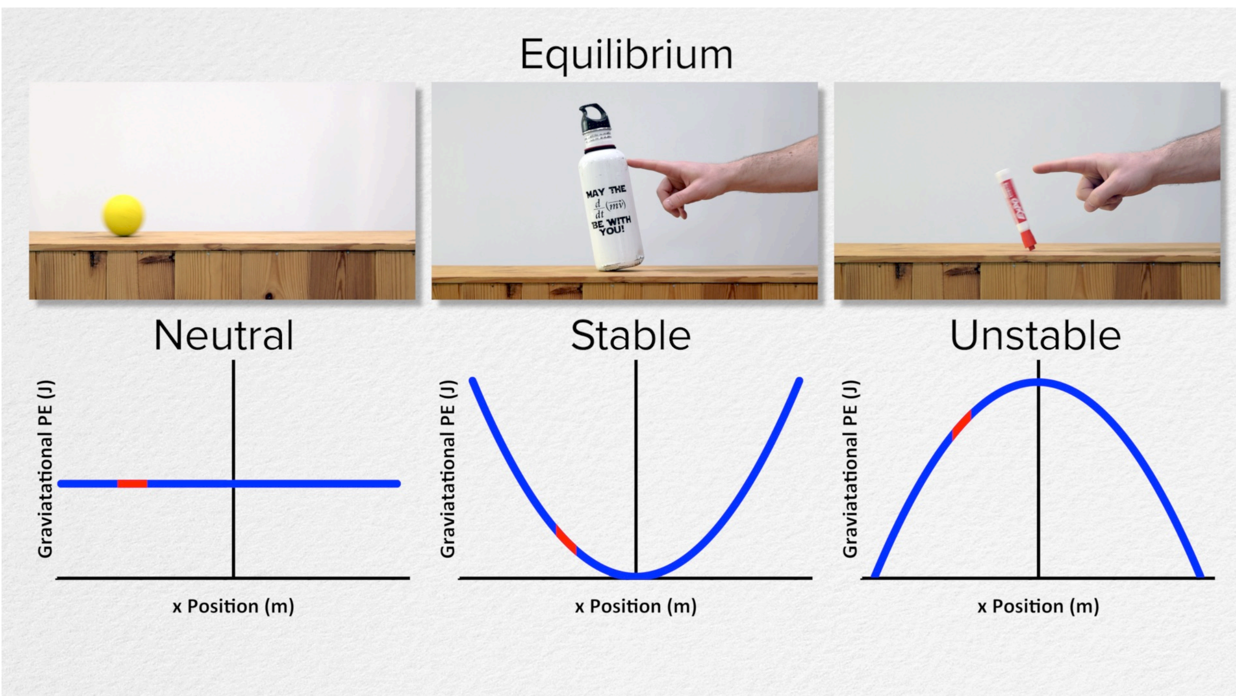
- Work done by a constant force: $W = \vec{F} \bullet \Delta\vec{r} = F\Delta r \cos\theta$
  - Work is the dot product of Force and the displacement of the object.
    - The dot product is also called the scalar product, because it is a scalar.
    - $W \Rightarrow joules,\ J = N \cdot m = \left(\frac{kg \cdot m}{s^2}\right)m$
  - Example: $\vec{F} = \left[2.7\hat{i} - 3.1\hat{j}\right]N$ and $\Delta\vec{r} = 4.6\hat{i}\ m$ then **(include drawing)**

    $W = \vec{F} \bullet \Delta\vec{r} = \left[2.7\hat{i} - 3.1\hat{j}\right] \bullet \left[4.6\hat{i} + 0\hat{j}\right] = (2.7)(4.6) + (-3.1)(0) = 12.42 \approx \boxed{12J}$
- Work done by a non-constant force: $W = \int_{x_i}^{x_f} F_x\,dx$
  - This is a definite integral.
    - "Definite" simply means it has limits x_i and x_f.
    - Integral, or anti-derivative is the area "under" the curve.
      - Area "under" the curve specifically means the area between the curve and the horizontal axis where area above the horizontal axis is positive and area below the horizontal axis is negative.
- Notice we now have two different equations for work, one for work done by a constant force and one fore work done by a force that varies. This will happen very often in AP Physics C and you need to be careful to identify the difference.
- The force caused by a spring: $\vec{F}_s = -k\Delta\vec{x}$
  - k is the "spring constant" and is a measure of how much force is takes to compress or expand a spring per meter.
  - Δx is the displacement of the spring from equilibrium position (or rest position).
  - The negative means the force of the spring is opposite the direction of the displacement of the spring.
- $W_{F_s} = \int_{x_i}^{x_f} F_s\,dx = \int_{x_i}^{x_f} (-kx)\,dx = \left(-\frac{kx^2}{2}\right)\Bigg]_{x_i}^{x_f} = \left(-\frac{kx_f^2}{2}\right) - \left(-\frac{kx_i^2}{2}\right)$

  $\Rightarrow W_{F_s} = -\left[\left(\frac{kx_f^2}{2}\right) - \left(\frac{kx_i^2}{2}\right)\right] = -\left[U_{ef} - U_{ei}\right] = -\Delta U_e \Rightarrow \boxed{W_{F_s} = -\Delta U_e}$
  - We have defined elastic potential energy: $U_e = \frac{1}{2}kx^2$.
  - The above example shows the work done by the force of the spring equals the negative of the change in elastic potential energy of the spring.
- $\sum W = \int_{x_i}^{x_f} \sum F\,dx \Rightarrow \sum W = \Delta KE$ The Net Work – Kinetic Energy Theorem.
  - Derivation is here: http://www.flippingphysics.com/wnet-ke.html
  - This equation is *always* true.
  - This equation is where kinetic energy is defined: $KE = \frac{1}{2}mv^2$

- Gravitational Potential Energy in a constant gravitational field is: $U_g = mgh$
  - h is the "vertical height above the horizontal zero line" and you have to always identify the horizontal zero line.
  - If you prefer the equation from the AP sheet, it is: $\Delta U_g = mg\Delta h$
    - The AP equation is the "change in" gravitational potential energy.
- Energy can be neither created nor destroyed, so in a non-isolated system the change in energy of the system equals the sum of the energy transferred to or from the system: $\Delta E_{system} = \sum T$
  - If the system is isolated, no energy is transferred into or out of the system: $\Delta E_{system} = 0$
    - The change in energy of the system is the change in mechanical energy of the system plus the change in internal energy of the system. $\Delta ME + \Delta E_{internal} = 0$
    - The change in internal energy of the system is done by nonconservative forces or friction. In other words, the energy which is dissipated by friction goes into the system as internal energy. $\Delta E_{internal} = -W_{nc}$
      - A nonconservative force is a force where the work done by the force *is* dependent on the path taken by the object. Conservative forces are where the work done by the force is *not* dependent on the path taken by the object.
    - In other words: $\Delta ME - W_{nc} = 0 \Rightarrow W_{nc} = \Delta ME$ and because I don't know of any forces which are nonconservative which are not friction:
    - $W_{friction} = \Delta ME$ (is only true when there is no energy transferred into our out of the system.)
  - If the system is isolated and there is no work done by friction:
    - $W_{friction} = \Delta ME \Rightarrow 0 = \Delta ME = ME_f - ME_i \Rightarrow ME_i = ME_f$
    - We have conservation of mechanical energy.
    - Which is only true when the system is isolated and no work is done by friction.
- Whenever you use $W_{friction} = \Delta ME$ or $ME_i = ME_f$ you have to identify the initial point, the final point and the horizontal zero line.
- All forms of Mechanical Energy are in terms of joules, just like Work.
- Power is the rate at which work is done: $P_{average} = \frac{W}{\Delta t}$ & $P_{instantaneous} = \frac{dW}{dt}$
  - $P_{instantaneous} = \frac{dW}{dt} = \frac{d}{dt}\left(\vec{F} \bullet \Delta\vec{r}\right) = F \bullet \frac{d\Delta\vec{r}}{dt} = \vec{F} \bullet \vec{v}$
    - Note: Force must be constant to use this equation.
  - The equations for power on the AP sheet are: $P = \frac{dE}{dt}$ & $P = \vec{F} \bullet \vec{v}$
  - $P \Rightarrow Watts = \frac{J}{s}$ & $746 watts = 1hp$
- Remember, every derivative is also an antiderivative (or an integral). For example:
  - $P = \frac{dW}{dt} \Rightarrow dW = P\,dt \Rightarrow \int_{W_i}^{W_f} dW = \int_{t_i}^{t_f} P\,dt \Rightarrow \Delta W = \int_{t_i}^{t_f} P\,dt$

- The equation which relates conservative forces and potential energy is: $F_x = -\frac{dU}{dx}$ (and it is *not* on the AP equation sheet.)
    - Aside: Much of the time when the phrase "conservative force" is used on the AP Exam, you need to use this equation.
    - For a spring: $F_s = -\frac{dU_e}{dx} = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -kx$
    - For gravity: $F_g = -\frac{dU_g}{dy} = -\frac{d}{dy}\left(mgy\right) = -mg$ (Force of gravity is always down)
- Neutral Equilibrium is where the Potential Energy of the object remains constant regardless of position. For example, a ball rolling on a level surface.
- Stable Equilibrium is where the Potential Energy of the object increases as the position of the object moves away from the equilibrium position and therefore the object naturally returns to the equilibrium position. For example, a water bottle being tipped to the side.
- Unstable Equilibrium is where the Potential Energy of the object decreases as the position of the object moves away from the equilibrium position and therefore the object naturally moves away from the equilibrium position. For example, a marker being tipped to the side.



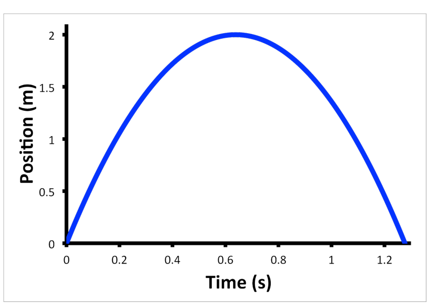
Flipping Physics Lecture Notes:

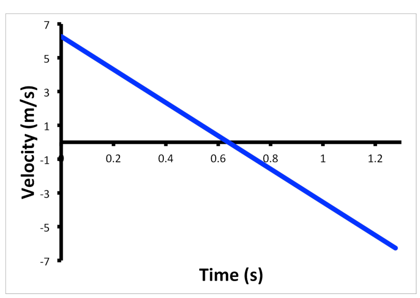
AP Physics C: Integrals in Kinematics Review (Mechanics)
https://www.flippingphysics.com/apc-integrals-kinematics-review.html

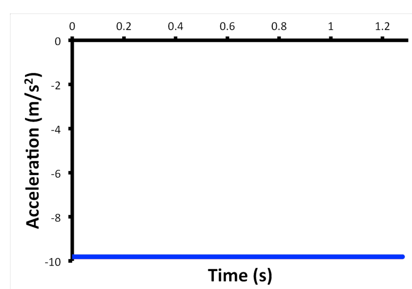
To be reviewed *after* students learn about integrals!!

- FYI: I do not teach integrals until we get to Work. By then the students who are taking calculus concurrently with AP Physics C Mechanics have had enough experience with derivatives that they only freak out a little bit when I teach them integrals. ☺

- Remember, every derivative is also an antiderivative (or an integral). For example:
  - $a=\frac{dv}{dt}\Rightarrow dv=a\,dt\Rightarrow\int_{v_i}^{v_f}dv=\int_{t_i}^{t_f}a\,dt\Rightarrow v\Big]_{v_i}^{v_f}=v_f-v_i=\Delta v=\int_{t_i}^{t_f}a\,dt$
  - The area "under" an acceleration as a function of time graph is the change in velocity of the object.
    - Remember the area "under" the curve specifically means the area between the curve and the horizontal axis where area above the horizontal axis is positive and area below the horizontal axis is negative.
- Another Example:
  - $v=\frac{dx}{dt}\Rightarrow dx=v\,dt\Rightarrow\int_{x_i}^{x_f}dx=\int_{t_i}^{t_f}v\,dt\Rightarrow x\Big]_{x_i}^{x_f}=x_f-x_i=\Delta x=\int_{t_i}^{t_f}v\,dt$
  - The area "under" an velocity as a function of time graph is the change in position of the object or the displacement of the object.
- Graphs of throwing a ball upward with a positive velocity initial.
  - $v=\frac{dx}{dt}$ → Velocity is the slope of a position vs. time graph.
  - $a=\frac{dv}{dt}$ → Acceleration is the slope of a velocity vs. time graph.
  - $\Delta v=\int_{t_i}^{t_f}a\,dt$ → Change in velocity is the area "under" an acceleration as a function of time graph.
  - $\Delta x=\int_{t_i}^{t_f}v\,dt$ → Change in position or displacement, is the area "under" a velocity as a function of time graph.







- Assuming the acceleration is constant, we can derive two of the Uniformly Accelerated Motion equations. For example:
  - $a = \frac{dv}{dt} \Rightarrow dv = a\,dt \Rightarrow \int dv = \int a\,dt \Rightarrow v(t) = at + C$
  - $\Rightarrow v(0) = a(0) + C \Rightarrow v(0) = C = v_i \Rightarrow v(t) = at + v_i \Rightarrow v_f = v_i + at$
- Another example:
  - $v = \frac{dx}{dt} \Rightarrow dx = v\,dt \Rightarrow \int dx = \int v\,dt \Rightarrow x(t) = \int (v_i + at)dt = v_i t + \frac{1}{2}at^2 + C$
  - $\Rightarrow x(0) = v_i(0) + \frac{1}{2}a(0)^2 + C \Rightarrow x(0) = C = x_i \Rightarrow x(t) = x_i + v_i t + \frac{1}{2}at^2$

Flipping Physics Lecture Notes:

AP Physics C: Momentum, Impulse, Collisions & Center of Mass Review (Mechanics)
https://www.flippingphysics.com/apc-momentum-impulse-review.html

- The symbol for momentum is a lowercase p. $\vec{p} = m\vec{v}$
    - Momentum is a vector!
    - $\vec{p} = m\vec{v} \Rightarrow \frac{kg \cdot m}{s}$ (not to be confused with Newtons which are $\frac{kg \cdot m}{s^2}$)
- Newton's 2nd law in terms of momentum is: $\sum \vec{F} = \frac{d\vec{p}}{dt}$
- $\sum \vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right) = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$ (the product rule)
    - $\Rightarrow \sum \vec{F} = \frac{dm}{dt}\vec{v} + m\vec{a}$ (Usually we assume the mass of the object does not change.)
- $\sum \vec{F}_{external} = \frac{d\vec{p}}{dt} = 0 \Rightarrow \sum \vec{p}_i = \sum \vec{p}_f$: When all the forces are internal to the system, the net force equals zero, the derivative of momentum as a function of time is zero, therefore the momentum does not change, therefore momentum is conserved.
    - Momentum is conserved during collisions and explosions.
- Impulse derivation: $\sum \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \sum \vec{F}dt = d\vec{p} \Rightarrow \int_{t_i}^{t_f} \sum \vec{F}\,dt = \int_{p_i}^{p_f} d\vec{p} = \vec{p}_f - \vec{p}_i \Rightarrow \Delta\vec{p} = \int_{t_i}^{t_f} \sum \vec{F}\,dt = \vec{J}$
    - Symbol for Impulse is J and it is a vector.
    - Units for Impulse: $\vec{J} = \int_{t_i}^{t_f} \sum \vec{F}\,dt \Rightarrow N \cdot s$ (yes, this is the same as momentum: $\frac{kg \cdot m}{s}$)
        - $N \cdot s = \left(\frac{kg \cdot m}{s^2}\right)s = \frac{kg \cdot m}{s}$
    - Impulse is the area "under" a force as a function of time curve.
        - Not to be confused with the equation for work: $W = \int_{x_i}^{x_f} F_x\,dx$
    - Impulse approximation says $\sum \vec{F} \approx \vec{F}_{impact}$
        - Therefore, the impulse approximation says: $\vec{F}_{impact} = \frac{d\vec{p}}{dt}$
        - Impulse, J, and Impact Force often get confused. Please note they are different!
    - Can also use the average force and change in time to determine Impulse: $\vec{J} = \vec{F}_{average}\Delta t$
        - This creates a rectangle with the same area as $\vec{J} = \int_{t_i}^{t_f} \sum \vec{F}\,dt$

Type of Collision	Is Momentum Conserved?	Is Kinetic Energy Conserved?
Elastic	Yes	Yes
Inelastic	Yes	No

- Collisions between hard spheres are "nearly" elastic and therefore are generally considered to be elastic in physics classes.
- "Perfectly Inelastic" Collisions are where the objects stick to one another. Sometimes they are called "Completely Inelastic" or "Totally Inelastic". These terms all mean the same thing.
- Most collisions are actually Inelastic.

- Center of mass of a system of particles: $x_{cm} = \dfrac{\sum m_i x_i}{\sum m_i} = \dfrac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots}$
  - x is the distance from a zero reference line; usually the origin.
  - Velocity of a system of particles: $v_{cm} = \dfrac{dx_{cm}}{dt} = \dfrac{d}{dt}\left(\dfrac{\sum m_i x_i}{\sum m_i}\right) = \dfrac{\sum m_i}{\sum m_i}\dfrac{dx_i}{dt} = \dfrac{\sum m_i v_i}{\sum m_i}$
  - Do the same thing with acceleration: $a_{cm} = \dfrac{dv_{cm}}{dt} = \dfrac{d}{dt}\left(\dfrac{\sum m_i v_i}{\sum m_i}\right) = \dfrac{\sum m_i a_i}{\sum m_i}$

- Center of mass of an object with shape: $r_{cm} = \dfrac{1}{m_{total}}\int r\,dm$ (not on AP equation sheet)
  - The position of the center of mass of an object with shape equals one over the total mass of the object times the integral with respect to mass of the posotion of all of the infinitesimally small pieces of the object, which are called dm, relative to a zero-reference line.
  - If you prefer: $x_{cm} = \dfrac{1}{m_{total}}\int x\,dm$
    - Volumetric Mass Density: $\rho = \dfrac{m}{\forall}$ (not on AP equation sheet)
    - Surface Mass Density: $\sigma = \dfrac{m}{A}$ (not on AP equation sheet)
    - Linear Mass Density: $\lambda = \dfrac{m}{L}$ (not on AP equation sheet)

Flipping Physics Lecture Notes:

AP Physics C: Rotational Kinematics Review (Mechanics)
https://www.flippingphysics.com/apc-rotational-kinematics-review.html

- Angular velocity: $\vec{\omega}_{average} = \frac{\Delta\vec{\theta}}{\Delta t}$ & $\vec{\omega}_{instantaneous} = \frac{d\vec{\theta}}{dt}\left(\frac{rad}{s} or \frac{rev}{\min}\right)$
- Angular acceleration: $\vec{\alpha}_{average} = \frac{\Delta\vec{\omega}}{\Delta t}$ & $\vec{\alpha}_{instantaneous} = \frac{d\vec{\omega}}{dt}\left(\frac{rad}{s^2}\right)$
- Uniformly Angularly Accelerated Motion: $U\alpha M$ (when $\alpha = constant = \#$)
  - 5 variables, 4 equations, If you know 3 variables, you can find the other 2, which leaves you with 1 …
  - $\omega_f = \omega_i + \alpha\Delta t\,; \Delta\theta = \omega_i\Delta t + \frac{1}{2}\alpha\Delta t^2;\ \ \omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta;\ \ \Delta\theta = \frac{1}{2}\left(\omega_i + \omega_f\right)\Delta t$
- Arc length, s, is the linear distance travelled when moving along a circle or part of a circle. In other words it is the linear length when traveling along an arc.
  - $s = r\Delta\theta$: arc length equals the radius of the object times the angular displacement of the object.
    - Must use radians when using this equation.
    - $1\ revolution = 360° = 2\pi\ radians$
    - The equation for circumference is an example of this equation where the angular displacement is one revolution or 2π radians: $C = r\left(2\pi\right)$
    - Arc length is a linear dimension, so its units are linear: meters, etc.
    - Not on equation sheet
    - I use a lowercase cursive $\mathscr{s}$ for arc length, because my s looks like a 5. Sorry.
- $s = r\Delta\theta \Rightarrow \frac{d}{dt}\left(s = r\Delta\theta\right) \Rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt} \Rightarrow v_t = r\omega$ (is on the AP equation sheet)
- $v_t = r\omega \Rightarrow \frac{d}{dt}\left(v_t = r\omega\right) \Rightarrow \frac{dv_t}{dt} = r\frac{d\omega}{dt} \Rightarrow a_t = r\alpha$ (not on the AP equation sheet)
  - Both of these equations assume the radius stays constant.
  - Must use radians when using both of these equations.
  - v_t is tangential velocity, or the linear velocity of an object moving in a circle. $\left(\frac{m}{s}\right)$
  - a_t is tangential acceleration, or the linear acceleration of an object moving in a circle. $\left(\frac{m}{s^2}\right)$
    - Both tangential quantities are tangent to the circle the object is moving along. This also means they are perpendicular to the radius of the circle the object is moving along.
- Uniform Circular Motion is where objects move in a circle with an angular acceleration of zero.
  - $\alpha = 0$ (The symbol for angular acceleration is alpha, α .)
  - Even though the magnitude of the object's velocity does not change, the direction of the velocity does, that means the velocity is not constant, therefore there must be an acceleration. The acceleration responsible for this change in the direction of the velocity is called centripetal acceleration, a_c.

- $a_c = \frac{v_t^2}{r} = r\omega^2$ in $\left(\frac{m}{s^2}\right)$
- Centripetal means "center seeking" because the centripetal acceleration is **always** toward the center of the circle the objects path describes.

- According to Newton's 2nd law, where there is an acceleration, there must be a net force. Therefore, if an object is moving in a circle, there is a centripetal acceleration and there must be a centripetal force.
  - Centripetal force: $\sum F_{in} = ma_c$
  - Centripetal force is the net force in the in direction.
    - It is not a new force.
    - It is never in a free body diagram
    - The "in" direction is positive. (The "out" direction is negative.)
  - See "conical pendulum" example from AP Physics 1 Kinematics Review.
- Non-Uniform Circular Motion will have an angular acceleration which is nonzero. $\alpha \neq 0$
  - This means there will also be a *tangential* acceleration, a_t, which is parallel to the tangential velocity and normal to the centripetal acceleration.
  - The net acceleration of an object in Non-Uniform Circular Motion is: $\vec{a}_{net} = \vec{a}_c + \vec{a}_t$
- The period, T, of an object moving in a circle is the time it takes for one revolution. Therefore:
  - $\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega}$: Period is in seconds

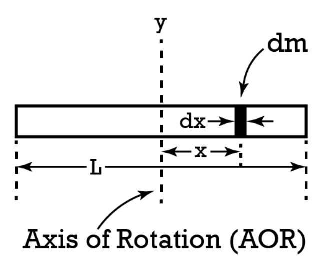
Flipping Physics Lecture Notes:

AP Physics C: Rotational Dynamics Review – 1 of 2 (Mechanics)
https://www.flippingphysics.com/apc-rotational-dynamics-1-review.html

- A rigid object with shape is rotating. Every piece of this object has kinetic energy. The total kinetic energy is the sum of all of the kinetic energies of every small piece of the object:

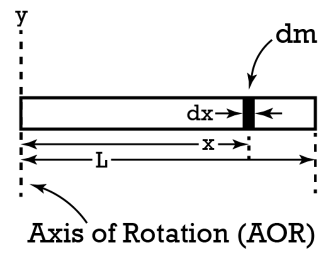
$$KE_t = \sum_i KE_i = \sum_i \frac{1}{2} m_i \left(v_i\right)^2 = \sum_i \frac{1}{2} m_i \left(r_i \omega_i\right)^2 = \sum_i \frac{1}{2} m_i r_i^2 \omega_i^2 = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I \omega^2$$

  - This uses $v_t = r\omega$ and that every part of the object has the same angular velocity, ω
  - $KE_{rotational} = \frac{1}{2} I \omega^2$: Rotational Kinetic Energy of a rigid object with shape or a system of particles that is not changing shape.
- $I = \sum_i m_i r_i^2$ where I is called the Moment of Inertia or "Rotational Mass".
  - This is the Moment of Inertia for a system of particles.
  - Units for Moment of Inertia: $I = \sum_i m_i r_i^2 \Rightarrow kg \cdot m^2$
- Moment of Inertia for a rigid object with shape: $I = \lim_{\Delta m \to 0} \sum r_i^2 \Delta m_i \Rightarrow I = \int r^2\, dm$
  - Not to be confused with the equation for the center of mass of a rigid object with shape: $r_{cm} = \frac{1}{m_{total}} \int r\, dm$
- Deriving the Moment of Inertia of a Uniform Thin Hoop about its Cylindrical Axis
  - $I_z = \int r^2\, dm = R^2 \int dm = R^2 m \Rightarrow I_{cm} = mR^2$
  - "Thin" means all of the dm's are located a distance R from the center of mass.
  - "Uniform" means the hoop is of a constant density.
  - "Cylindrical Axis" means the line through the center of the hoop and normal to the plane of the hoop.
- Deriving the Moment of Inertia of a Uniform Rigid Rod about its Center of Mass



  - $\lambda = \frac{m}{L} = \frac{dm}{dx} \Rightarrow dm = \lambda\, dx \Rightarrow dm = \frac{m}{L}\, dx$
    - m is the total mass of the rod
    - L is the total length of the rod
    - λ is the linear mass density of the rod, which is constant in this "uniform" rod.
  - $I_y = \int r^2\, dm = \int r^2 \frac{m}{L} dx = \frac{m}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\, dx = \frac{m}{L}\left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}}$
  - $\Rightarrow I_y = \frac{m}{L}\left[\frac{\left(\frac{L}{2}\right)^3}{3} - \frac{\left(-\frac{L}{2}\right)^3}{3}\right] = \frac{m}{L}\left[\frac{L^3}{24} + \frac{L^3}{24}\right] = \frac{m}{L}\left[\frac{2L^3}{24}\right] = \boxed{\frac{1}{12} mL^2}$

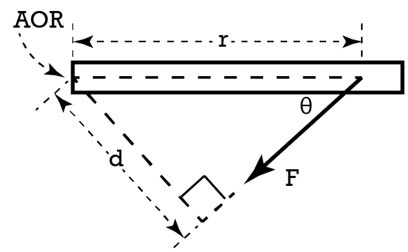
- Deriving the Moment of Inertia of a Uniform Rigid Rod about one end
    - This is the same as before, only with different limits …
    - $I_y = \frac{m}{L}\int_0^L x^2\,dx \Rightarrow \frac{m}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{m}{L}\frac{L^3}{3} = \boxed{\frac{1}{3}mL^2}$



- The Parallel-Axis Theorem: $I = I_{cm} + mD^2$
    - Only true for objects with constant density.
    - m is the total mass of the rigid, constant density object.
    - D is the distance from the center of mass of the object to the new axis of rotation.
    - Not on the AP equation sheet.
- Example: Moment of Inertia of a Uniform Rigid Rod about its end.
    - Known for Uniform Rigid Rod: $I_{cm} = \frac{1}{12}mL^2$

$$I_{end} = I_{cm} + mD^2 = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{12}mL^2 + \frac{1}{4}mL^2 = \left(\frac{1}{12} + \frac{3}{12}\right)mL^2 = \frac{4}{12}mL^2 = \boxed{\frac{1}{3}mL^2}$$

- Example: Moment of Inertia of a Uniform Thin Hoop about its Rim.
    - Known for Uniform Thin Hoop about its Center of Mass: $I_{cm} = mR^2$
    - $I_{rim} = I_{cm} + mD^2 = mR^2 + mR^2 = \boxed{2mR^2}$
- Torque: $\tau = rF\sin\theta$
    - This is the magnitude of the torque. Torque is a vector.
    - r is the distance from the axis of rotation to the location on the object the force is applied.
    - F is the magnitude of the force.
    - θ is the angle between r and F.
    - $\sin\theta = \frac{O}{H} = \frac{d}{r} \Rightarrow d = r\sin\theta$ is the "moment arm" or "lever arm" or "effective distance"
    - Units for torque are $N \cdot m$
        - Not to be confused with the units for energy, joules, even though joules are also $N \cdot m$.



- But "What is torque?" Torque is the rotational equivalent of force. Force is the ability to cause an acceleration of an object. Torque is the ability of a force to cause an *angular* acceleration of an object.
- The rotational form of Newton's Second Law: $\sum \vec{F} = m\vec{a} \Rightarrow \sum \vec{\tau} = I\vec{\alpha}$
    - Must identify axis of rotation when summing the torques.
    - Must identify what objects you are summing the torque on.
        - Note: The angular acceleration of each object around the axis of rotation must be the same.
    - Must identify the direction of positive rotation.
    - Now that we have defined Moment of Inertia, pulleys can have mass. When pulleys have mass the force of tension on either side of a pulley are *not* the same!
- Right Hand Rule for direction of torque
    - Don't be too cool for the right hand rule. Limber Up!
    - Use your right hand.
    - Fingers start at the axis of rotation.
    - Point fingers along direction of "r".
    - Curl fingers along the direction of "F".
    - Thumb points in the direction of the torque.

- Rolling Without Slipping: $v_{cm} = R\omega$ & $a_{cm} = R\alpha$
    - Just like $v_t = r\omega$ & $a_t = r\alpha$ only …
        - R is the radius of the solid object
        - These are for the center of mass of the object, not the tangential quantities.
    - Neither of these are on the AP equation sheet.
    - FYI: Rolling *With* Slipping: $v_{cm} \neq R\omega$ & $a_{cm} \neq R\alpha$
    - When an object is rolling without slipping it has **both** translational and rotational kinetic energies!!

Flipping Physics Lecture Notes:
AP Physics C: Rotational Dynamics Review – 2 of 2 (Mechanics)
https://www.flippingphysics.com/apc-rotational-dynamics-2-review.html

- $\vec{\tau} = \vec{r} \times \vec{F} \neq \vec{F} \times \vec{r}$
    - Torque is the cross product (also called the vector product) of $\vec{r}$ & $\vec{F}$.
        - Torque is a vector!
    - $\vec{r}$ is the position vector from the axis of rotation to the location of the force, $\vec{F}$.
    - Magnitude of torque → $\tau = rF\sin\theta$
    - The order does matter! ($\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$)
    - Cross product is the area of the parallelogram with sides $\vec{r}$ & $\vec{F}$.
- In case you forgot how to do the cross product. Example: $\vec{A} = -\hat{i} + \hat{j} - 2\hat{k}$ & $\vec{B} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 2 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} \hat{i} - \begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} \hat{j} + \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix} \hat{k}$$

$$\Rightarrow \vec{A} \times \vec{B} = \left[(1)(4) - (-2)(-3)\right]\hat{i} - \left[(-1)(4) - (-2)(2)\right]\hat{j} + \left[(-1)(-3) - (1)(2)\right]\hat{k}$$

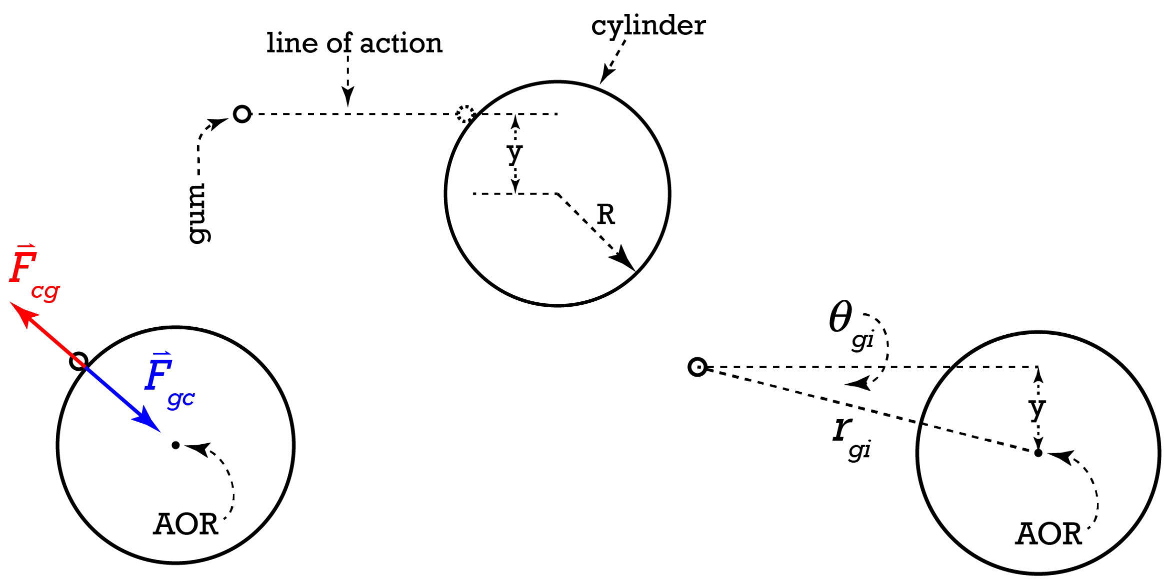
$$\Rightarrow \vec{A} \times \vec{B} = \left[4 - 6\right]\hat{i} - \left[-4 + 4\right]\hat{j} + \left[3 - 2\right]\hat{k} = \boxed{-2\hat{i} + \hat{k}}$$

- An object is in *Translational* Equilibrium if the net force acting on it equals zero, which means the object is not accelerating: $\sum \vec{F} = 0 = m\vec{a} \Rightarrow \vec{a} = 0$
- An object is in *Rotational* Equilibrium if the net torque acting on it equals zero, which means the object is not *angularly* accelerating: $\sum \vec{\tau} = 0 = I\vec{\alpha} \Rightarrow \vec{\alpha} = 0$ (must identify axis of rotation)
    - This means the object is either not rotating or has a constant angular velocity.
    - If an object is in translational equilibrium and in rotational equilibrium about *one* axis of rotation, then the object is in rotational equilibrium about *any* axis of rotation.
- $\vec{L}$ is Angular Momentum and it is a vector!
    - $\sum \vec{F} = m\vec{a} \Rightarrow \sum \vec{\tau} = I\vec{\alpha}$ & $\sum \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \sum \vec{\tau} = \frac{d\vec{L}}{dt}$
- For a *particle* or any object which is *not rotating*:
    - Just like torque, we have a cross product equation for angular momentum: $\vec{L} = \vec{r} \times \vec{p}$
        - r is the position vector from the axis of rotation to the location of the center of mass of the moving object.
        - And a magnitude equation for angular momentum: $L = rmv\sin\theta$
            - With this equation, need to use Right Hand Rule to find direction.
    - Yes, a particle or a rigid object which is not rotating can have an angular momentum!
- For a *rigid object with shape*: $\vec{L} = I\vec{\omega}$
    - Units for angular momentum: $\vec{L} = I\vec{\omega} \Rightarrow (kg \cdot m^2)\left(\frac{rad}{s}\right) = \frac{kg \cdot m^2 \cdot rad}{s} = \frac{kg \cdot m^2}{s}$
- Derivation of conservation of *linear* momentum: $\sum \vec{F}_{external} = \frac{d\vec{p}}{dt} = 0 \Rightarrow \sum \vec{p}_i = \sum \vec{p}_f$
- Derivation of conservation of *angular* momentum: $\sum \vec{\tau}_{external} = \frac{d\vec{L}}{dt} = 0 \Rightarrow \sum \vec{L}_i = \sum \vec{L}_f$

- Note the similarities between the two, please.
- Remember net torque requires the axis of rotation to be identified, which means the axis of rotation needs to be identified for conservation of angular momentum

- Conservation of Angular Momentum Example: A piece of gum with mass, m, and velocity, v, is spat at a solid cylinder of mass, M, radius, R, and moment of inertia $\frac{1}{2}MR^2$. The cylinder is on a horizontal axis through its center of mass and is initially at rest. The line of action of the gum is located horizontally a height, y, above the axis of the cylinder. If the gum sticks to the cylinder, what is the final angular velocity of the gum/cylinder system? The **Drawing!!**
  - Gum knowns: $m = m_g, v = v_{gi}$ Cylinder knowns: $M = m_c, \omega_{ic} = 0, R, I_c = \frac{1}{2}m_c R^2$
  - Solving for $\omega_f = ?$ (will be the same for both gum and cylinder)
  - Know angular momentum is conserved because: $\sum \vec{\tau}_{external} = \frac{d\vec{L}}{dt} = 0$
  - $\sum \vec{L}_i = \sum \vec{L}_f \Rightarrow \vec{L}_{gi} + \vec{L}_{ci} = \vec{L}_{gf} + \vec{L}_{cf} \Rightarrow r_{gi} m_g v_{gi} \sin\theta_{gi} + 0 = r_{gf} m_g v_{gf} \sin\theta_{gf} + I_c \omega_f$
  - $\sin\theta_{gi} = \frac{O}{H} = \frac{y}{r_{gi}} \Rightarrow y = r_{gi}\sin\theta_{gi} \Rightarrow y m_g v_{gi} = r_{gf} m_g v_{gf} \sin\theta_{gf} + I_c\omega_f$
  - $v_{gf} = v_t = R\omega_f \Rightarrow y m_g v_{gi} = R m_g R \omega_f \sin 90 + \frac{1}{2} m_c R^2 \omega_f$
  - $\Rightarrow y m_g v_{gi} = R^2 \omega_f \left(m_g + \frac{m_c}{2} \right) \Rightarrow \boxed{\omega_f = \frac{y m_g v_{gi}}{R^2 \left(m_g + \frac{m_c}{2} \right)}}$

FYI: Sawdog, one of my Quality Control Team members, pointed out that, after colliding with the cylinder, the gum is moving in a circle, so it's angular momentum can be described using $I_g \omega_f$. More specifically: $\vec{L}_{gf} = I_g \omega_f = \left(m_g r_g^{\,2} \right) \omega_f = m_g R^2 \omega_f$ It's a slightly different solution that results in the same answer.



Flipping Physics Lecture Notes:
AP Physics C: Rotational vs. Linear Review (Mechanics)
https://www.flippingphysics.com/apc-rotational-vs-linear-review.html

Name:	Linear:	Rotational:
Displacement	$\Delta\vec{x} = x_f - x_i$	$\Delta\vec{\theta} = \theta_f - \theta_i$
Velocity	$\vec{v}_{avg} = \frac{\Delta\vec{x}}{\Delta t}$ & $\vec{v}_{inst} = \frac{d\vec{x}}{dt}$	$\vec{\omega}_{avg} = \frac{\Delta\vec{\theta}}{\Delta t}$ & $\vec{\omega}_{inst} = \frac{d\vec{\theta}}{dt}$
Acceleration	$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$ & $\vec{a}_{inst} = \frac{d\vec{v}}{dt}$	$\vec{\alpha}_{avg} = \frac{\Delta\vec{\omega}}{\Delta t}$ & $\vec{\alpha}_{inst} = \frac{d\vec{\omega}}{dt}$
Uniformly Accelerated Motion (UAM) or Uniformly Angularly Accelerated Motion (UαM)	$v_f = v_i + at$ $x_f = x_i + v_i t + \frac{1}{2}at^2$ $v_f^2 = v_i^2 + 2a(x_f - x_i)$ $x_f - x_i = \frac{1}{2}(v_f + v_i)t$	$\omega_f = \omega_i + \alpha t$ $\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$ $\omega_f = \omega_i + 2\alpha(\theta_f - \theta_i)$ $\theta_f - \theta_i = \frac{1}{2}(\omega_f + \omega_i)t$
Mass	Mass	$I_{particles} = \sum_i m_i r_i^2$ $I_{object\ with\ shape} = \int r^2\,dm$
Kinetic Energy	$KE_{translational} = \frac{1}{2}mv^2$	$KE_{rotational} = \frac{1}{2}I\omega^2$
Newton's Second Law	$\sum\vec{F} = m\vec{a}$ & $\sum\vec{F} = \frac{d\vec{p}}{dt}$	$\sum\vec{\tau} = I\vec{\alpha}$ & $\sum\vec{\tau} = \frac{d\vec{L}}{dt}$
Force / Torque	Force	$\vec{\tau} = \vec{r} \times \vec{F}$
Power	$P_{translational} = \frac{dW}{dt} = \vec{F} \bullet \vec{v}$	$P_{rotational} = \frac{dW}{dt} = \vec{\tau} \bullet \vec{\omega}$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L}_{particle} = \vec{r} \times \vec{p}$ $\vec{L}_{object\ with\ shape} = I\vec{\omega}$

Thank you to Aarti Sangwan for pointing out that I didn't include a rotational form of work in the video.

Name:	Linear:	Rotational:
Work (constant force)	$W = \vec{F} \cdot \Delta\vec{r} = F\Delta r\cos\theta$	$W = \vec{\tau} \cdot \Delta\vec{\theta}$
Work (non-constant force)	$W = \int_{x_i}^{x_f} F_x\,dx$	$W = \int_{\theta_i}^{\theta_f} \tau\,d\theta$
Net Work-Kinetic Energy Theorem	$W_{net} = \Delta KE = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$	$W_{net} = \Delta KE = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$

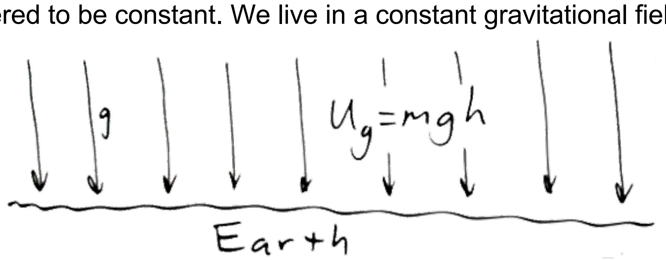
A little bonus: Look what happens when we combine a couple of the above formulas:

$$W_{net} = \vec{\tau}_{net} \cdot \Delta\vec{\theta} = I\alpha\Delta\theta = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2 \Rightarrow 2\alpha\Delta\theta = \omega_f^2 - \omega_i^2 \Rightarrow \omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta \text{ (UαM!)}$$

Flipping Physics Lecture Notes:

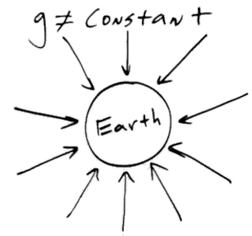
AP Physics C: Universal Gravitation Review (Mechanics)
https://www.flippingphysics.com/apc-universal-gravitation-review.html

- The Force of Gravity or Weight of an object: $F_g = mg$
  - A subscript is missing: $F_g = m_o g$ where "o" stands for "object".
  - All forces require two objects. This equation is the force of gravitational attraction which exists between the object and a planet.
    - For us, the planet usually is the Earth. $g_{Earth} = 9.81\frac{m}{s^2}$
- Any two objects which have mass have a force of gravitational attraction between them. This force is determined using Newton's Universal Law of Gravitation. (The Big G Equation)
  - $F_g = \frac{Gm_1m_2}{r^2}$: G is the Universal Gravitational Constant. $G = 6.67\times10^{-11}\frac{N\cdot m^2}{kg^2}$
    - Requires ***two*** objects: mass 1 and mass 2.
    - r is ***not*** the radius by definition. r is the distance between the centers of mass of the two objects. Yes, sometimes r is the radius.
    - This equation is the magnitude of the force of gravitational attraction. The force is always directed toward the other object.
  - Thank you to one of my Quality Control team members, Frank Geshwind, for pointing out that I should have talked about the vector form of this equation: $\vec{F}_g = -\frac{Gm_1m_2}{r^2}\hat{r}_{12}$
    - Note: $\hat{r}_{12}$ is the unit vector from object 1 to object 2.
    - Some textbooks even use this equation: $\vec{F}_g = -\frac{Gm_1m_2}{r^3}\vec{r}_{12}$
      - Note the subtle change here. $\vec{r}_{12}$ is no longer the unit vector and therefore the cube of r needs to be in the denominator. ☺
- Setting the two equations for the Force of Gravity acting on an object on the surface of the Earth equal to one another yields this:
  - $F_g = F_g \Rightarrow m_o g = \frac{Gm_o m_E}{r^2} \Rightarrow g_{Earth} = \frac{Gm_E}{\left(R_{Earth} + Altitude\right)^2}$
- As long as $R_{Earth} \gg \Delta Altitude$ the $g_{Earth} \approx constant$.
  - In other words, close to the surface of the planet, the acceleration due to gravity can be considered to be constant. We live in a constant gravitational field.

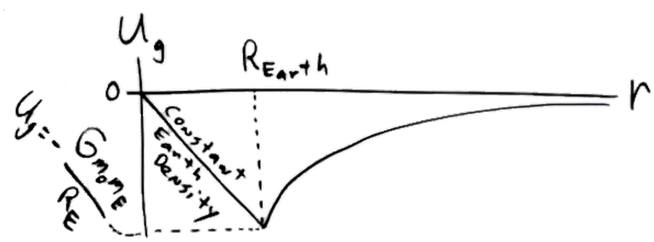


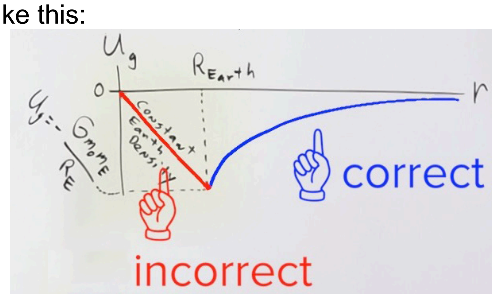
  - The gravitational potential energy is then constant: $U_g = mgh$

- When viewed from a frame of reference which is not on the surface of the planet, the acceleration due to gravity is *not* constant and we need a different equation:
  - Universal Gravitational Potential Energy: $U_g = -\frac{Gm_1m_2}{r}$
    - This equation assumes a zero line which is infinitely far away. ($r \approx \infty$)
    - This means Universal Gravitational Potential Energy *can never be positive.*
    - This equation requires two objects.
    - Note: This equation is not $F_g = \frac{Gm_1m_2}{r^2}$ r is *not* squared.



- The gravitational potential energy which exists between an object and the Earth in terms of the objects distance, r, from the center of the Earth looks like this:





  - Assumes constant density Earth. (Which is not true, however, we often assume it is.)
  - Be aware the graph as shown above left has an incorrect part:
    - This cannot be correct because this implies an object would experience a change in gravitational potential energy of zero.
      - $\Delta U_g = U_f - U_i = 0 - 0 = 0$ (This makes no sense.)
    - This implies it takes zero energy to move an object from the center of a planet to infinitely far from the planet.
      - $W_{F_a} = \Delta ME = \Delta U_g = 0$ (Again, this makes no sense.)
    - This would mean the planet has a binding energy of zero; still makes no sense.
    - I will be releasing a video with the correct solution soon.
  - The Gravitational Potential Energy which exists between the two objects when the object is on the surface of the Earth is: $U_g = -\frac{Gm_om_E}{R_E}$
- What is the minimum amount of work necessary to completely remove an object from a planet if the object is resting on the surface of the planet? This is called the Binding Energy. Assume the object is moved infinitely far away, has zero velocity when it gets there, and there is no friction.

$$\Delta E_{system} = \sum T \Rightarrow \Delta ME + \Delta E_{\text{internal}} = W_{F_a}$$

$$\Rightarrow W_{F_a} = ME_f - ME_i + 0 = 0 - U_{gi} = 0 - \left(-\frac{Gm_1m_2}{r}\right) = \frac{Gm_om_p}{R_p} \text{ (Binding Energy)}$$

- What is the minimum velocity to launch an object off the Earth and have it never return? This is called Escape Velocity. Assume no atmosphere and no Earth rotation. Note: Mechanical Energy is conserved:

$$ME_i = ME_f \Rightarrow U_{gi} + KE_i = 0 \Rightarrow -\frac{Gm_om_E}{R_E} + \frac{1}{2}m_ov_i^2 = 0 \Rightarrow \frac{1}{2}v_i^2 = \frac{Gm_E}{R_E} \Rightarrow v_{escape} = \sqrt{\frac{2Gm_E}{R_E}}$$

- What is the total mechanical energy of an object in circular orbit?

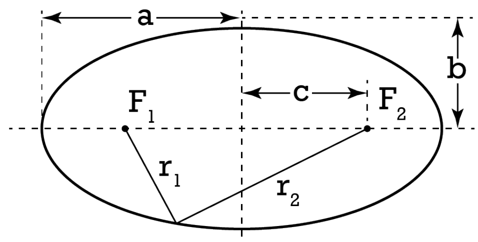
$$ME_{total} = U_g + KE = -\frac{Gm_o m_{planet}}{r} + \frac{1}{2} m_o v_o^{\ 2}$$

The only force acting on the object moving in circular orbit is the force of gravity which acts inward. So we can sum the forces on the object in the in direction:

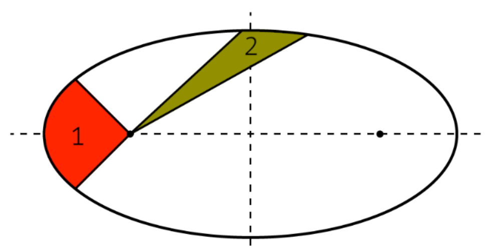
$$\sum F_{in} = F_g = ma_c \Rightarrow \frac{Gm_o m_p}{r^2} = m_o \frac{v_o^{\ 2}}{r} \Rightarrow \frac{Gm_o m_p}{2r} = \frac{1}{2} m_o v_o^{\ 2} = KE_o$$

$$ME_{total} = -\frac{Gm_o m_p}{r} + \frac{Gm_o m_p}{2r} = \frac{Gm_o m_p}{r}\left(-1 + \frac{1}{2}\right) = \frac{Gm_o m_p}{r}\left(-\frac{1}{2}\right) = -\frac{Gm_o m_p}{2r}$$

- Kepler's 3 laws. I find having a basic understanding of the his first two laws is adequate, however, you need to know how to derive Kepler's 3rd law.
- Kepler's 1st Law is that orbits are elliptical and defined as such:



  - Two foci at F_1 and F_2.
  - Each focus is located a distance c from the center of the ellipse.
  - Semimajor axis of length a. (Major axis of length 2a)
  - Semiminor axis of length b. (Minor axis of length 2b)
  - $r_1 + r_2 = 2a$
  - $a^2 = b^2 + c^2$
  - Eccentricity of an ellipse is defined as c/a. For a circle c = 0 therefore eccentricity = 0.
  - The eccentricity of the Earth's orbit is 0.017, which means its orbit is nearly circular.
- Kepler's 2nd Law states that a line between the sun and the planet sweeps out an equal area over an equal time interval.



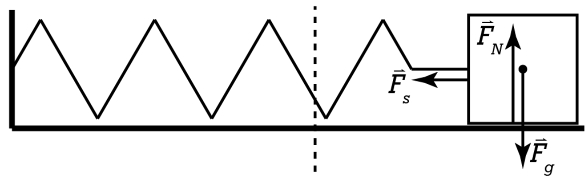
  - The result of this is that the closer an object is to the planet during an orbit, the faster its orbital speed will be.
- Let's derive Kepler's 3rd law: We assume circular orbit. The only force acting on the orbital object is the force of gravity acting inward. Sum the forces in the in direction:

$$\sum F_{in} = F_g = ma_c \Rightarrow \frac{Gm_o m_p}{r^2} = m_o r\omega^2 \Rightarrow \frac{Gm_p}{r^3} = \omega^2 = \left(\frac{\Delta\theta}{\Delta t}\right)^2 = \left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2}{T^2} \Rightarrow T^2 = \left(\frac{4\pi^2}{Gm_p}\right) r^3$$

Flipping Physics Lecture Notes:

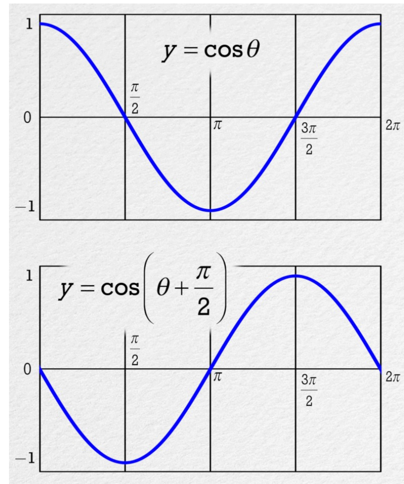
AP Physics C: Simple Harmonic Motion Review (Mechanics)
https://www.flippingphysics.com/apc-simple-harmonic-motion-review.html

- An object is in Simple Harmonic Motion if the acceleration of the object is proportional to the object's displacement from an equilibrium position and that acceleration is directed toward the equilibrium position. $a \propto \Delta x$
- For example: A horizontal mass-spring system on a frictionless surface has the following free body diagram:

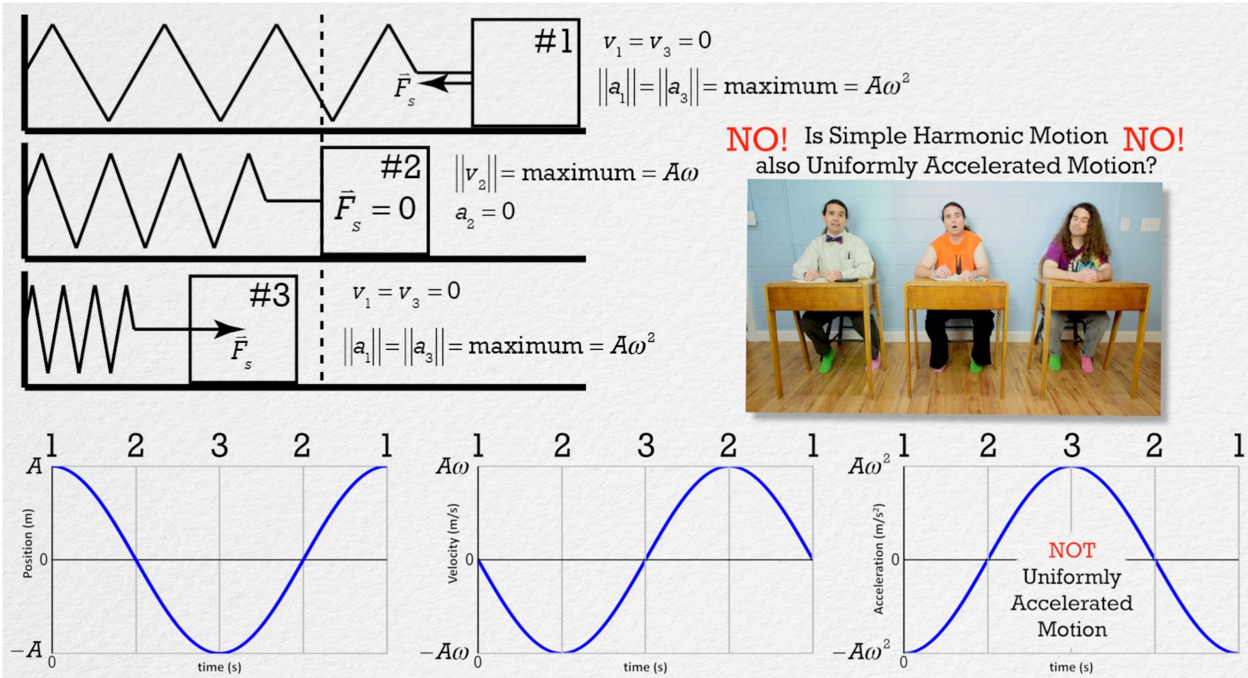


  - $\sum F_x = -F_s = ma_x \Rightarrow -kx = ma_x \Rightarrow a_x = -\frac{k}{m}x$
  - Amplitude, A, is defined as the maximum distance from equilibrium position. Therefore:
    - $a_{\max} = \frac{k}{m}A$
  - Note: $a = \frac{dv}{dt}$ & $v = \frac{dx}{dt} \Rightarrow a = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$
  - Therefore: $\frac{d^2x}{dt^2} = -\frac{k}{m}x$
  - Let $\frac{k}{m} = \omega^2$ where ω is called the angular frequency
  - Therefore: $\frac{d^2x}{dt^2} = -\omega^2 x$
    - This is the condition for simple harmonic motion.
    - This equation is *not* on the AP equation sheet. **Memorize It!!**
  - Note: $\omega = \sqrt{\frac{k}{m}} = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$ (The period of a mass-spring system)
  - Period of a pendulum: $T = 2\pi\sqrt{\frac{L}{g}}$ (know how to derive)
  - $T = \frac{1}{f}$ & $\omega = \frac{2\pi}{T} = 2\pi f \Rightarrow \omega = 2\pi f$
    - Frequency, f, is the number of cycles an object goes through per second.
    - Angular frequency and frequency are related, $\omega = 2\pi f$, however, they are not the same.

- One equation that satisfies the condition for Simple Harmonic Motion is: $x(t) = A\cos(\omega t + \phi)$
    - This equation *is* on the AP physics equation sheet, however, the equations for velocity and acceleration in simple harmonic motion are **not**.
    - Have to use angles in radians in this equation.
    - ϕ or "phi" is the "phase constant" or the "phase shift" of the wave. For example:
        - $y = \cos\left(\theta + \frac{\pi}{2}\right)$ is "phase shifted" to the left from $y = \cos\theta$ by $\frac{\pi}{2}$ radians.



- $v = \frac{dx}{dt} = \frac{d}{dt}\left(A\cos(\omega t + \phi)\right) = A\left(-\sin(\omega t + \phi)\right)(\omega)$
    - $\Rightarrow v(t) = A\frac{d}{dt}\left(\cos(\omega t + \phi)\right) = A\left(-\sin(\omega t + \phi)\right)\left[\frac{d}{dt}(\omega t + \phi)\right]$
    - $\Rightarrow v(t) = -A\omega\sin(\omega t + \phi)$
    - & $v_{max} = A\omega$
- $a = \frac{dv}{dt} = \frac{d}{dt}\left(-A\omega\sin(\omega t + \phi)\right) = -A\omega\frac{d}{dt}\left(\sin(\omega t + \phi)\right) = -A\omega\cos(\omega t + \phi)\left[\frac{d}{dt}(\omega t + \phi)\right]$
    - $\Rightarrow a = -A\omega\left(\cos(\omega t + \phi)\right)(\omega) \Rightarrow a(t) = -A\omega^2\cos(\omega t + \phi)$
    - & $a_{max} = A\omega^2$
- $\Rightarrow a(t) = -\omega^2\left(A\cos(\omega t + \phi)\right) = -\omega^2 x(t) \Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x$



- Simple Harmonic Motion is **NOT** Uniformly Accelerated Motion

- Total mechanical energy in Simple Harmonic Motion:
  - $ME_{total} = \frac{1}{2}kA^2 = \frac{1}{2}m\left(v_{\max}\right)^2$

Flipping Physics Lecture Notes:

AP Physics C: Equations to Memorize (Mechanics)
https://www.flippingphysics.com/apc-equations-to-memorize.html

While I am not a fan of memorization and do my best to avoid having my students memorize, there are a few items which are not on the equation sheet which I do suggest you memorize.

- $\Delta x = \frac{1}{2}\left(v_f + v_i\right)\Delta t$ (The fourth Uniformly Accelerated Motion equation)
- The Force of Gravity or Weight of an object: $F_g = mg$
- $F_{g_\perp} = mg\cos\theta$ & $F_{g_\parallel} = mg\sin\theta$ (The components of the force of gravity parallel and perpendicular on an incline where θ is the incline angle)
- General range for coefficients of friction: 0 – 2
- $\Delta E_{system} = \sum T$ (General equation relating the change in energy of the system to the net energy transferred into or out of the system.)
- $\sum W = \Delta KE$ (always true)
- $W_{friction} = \Delta ME$ (only true when there is no energy added to or removed from the system via a force.)
- $ME_i = ME_f$ (only true when there is no energy added to or removed from the system via a force and there is no work done by a nonconservative force.)
- $F_x = -\frac{dU}{dx}$ (The equation which relates a **conservative** force and the potential energy associated with that force.)
- That every derivative is an integral and every integral is a derivative. For Example:
  - $F_x = -\frac{dU}{dx} \Rightarrow F_x dx = -dU \Rightarrow \int_{x_i}^{x_f} F_x\, dx = -\int_{U_i}^{U_f} dU \Rightarrow W = -\Delta U$
- Book Example: $W_{F_g} = F_g \Delta r\cos\theta = \left(mg\right)\Delta h\cos\left(180^\circ\right) = -mg\Delta h \Rightarrow W_{F_g} = -\Delta U_g$
- $\sum \vec{p}_i = \sum \vec{p}_f$ (Conservation of Momentum. It may seem obvious, however, you need to remember when it is valid.)
  - $\sum \vec{F}_{external} = \frac{d\vec{p}}{dt} = 0 \Rightarrow \sum \vec{p}_i = \sum \vec{p}_f$
- $\sum \vec{L}_i = \sum \vec{L}_f$ (Conservation of Angular Momentum. Again, it may seem obvious, however, you need to remember when it is valid.)
  - $\sum \vec{\tau}_{external} = \frac{d\vec{L}}{dt} = 0 \Rightarrow \sum \vec{L}_i = \sum \vec{L}_f$

- $r_{cm} = \frac{1}{m_{total}}\int r\, dm$ (The center of mass of a rigid object with shape)
- $\rho = \frac{m}{\forall}$ & $\lambda = \frac{m}{L}$ (Volumetric Mass Density and Linear Mass Density)

- $s = r\Delta\theta$ & $a_t = r\alpha$ (arc length and tangential acceleration)
  - Although $v_t = r\omega$ is on the equation sheet, so it is easy to get to the other two.
- $v_{cm} = R\omega$ & $a_{cm} = R\alpha$ (The velocity and acceleration of the center of mass of a rigid object which is rolling without slipping. Easy to remember from the previous equations.)
- $1\ revolution = 360^\circ = 2\pi\ radians$
- $\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta;\ \Delta\theta = \frac{1}{2}\left(\omega_i + \omega_f\right)\Delta t$ (Uniformly Angularly Accelerated Motion equations)
- $I = I_{cm} + mD^2$ (The parallel axis theorem)
- $\frac{d^2x}{dt^2} = -\omega^2 x$ (The condition for simple harmonic motion)
- $v_{\max} = A\omega$ (The maximum velocity during simple harmonic motion)
- $a_{\max} = A\omega^2$ (the maximum acceleration during simple harmonic motion)

There are equations which many of you will want to memorize, however, I strongly discourage.

- $\vec{F}_R = -b\vec{v}$ & $\vec{F}_R = \frac{1}{2}D\rho A v^2$ (Resistive force equations)
  - Neither of these equations are on the equation sheet. Don't memorize these equations.
    - The problem will specify to use $\vec{F}_R = \frac{1}{2}D\rho A v^2$ and give you that equation or tell you the drag force is "proportional to" the velocity, which means $\vec{F}_R = -b\vec{v}$.
- 746 watts = 1 hp (will be provided if you need it)
- Do not memorize: $G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$
  - Instead, be familiar with the "Table of Information" and the page of general math formulas on the AP Physics equation sheet.
- Do not memorize the following equations; instead know how to derive them. This will be of much more use to you during the AP exam because they want you to understand where these equations come from and therefore will generally ask you a question that relates to their derivations. I did all of these derivations during the review.
  - $v_{cm} = \frac{\sum m_i v_i}{\sum m_i}$ & $a_{cm} = \frac{\sum m_i a_i}{\sum m_i}$ (velocity and acceleration of the center of mass of a system of particles. Simply take the derivative with respect to time once or twice of the position of the center of mass of a system of particles to get these equations.)
  - Terminal velocity: $v_{terminal} = \sqrt{\frac{2mg}{D\rho A}}$
  - Binding Energy: $W_{F_a} = \frac{G m_o m_p}{R_p}$
  - Escape Velocity: $v_{escape} = \sqrt{\frac{2Gm_E}{R_E}}$

- Total Mechanical Energy of Orbital Object: $ME_{total} = -\frac{Gm_o m_p}{2r}$
- Kepler's Third Law: $T^2 = \left(\frac{4\pi^2}{Gm_p}\right) r^3$
- Velocity in simple harmonic motion: $v(t) = -A\omega \sin(\omega t + \phi)$
- Acceleration in simple harmonic motion: $a(t) = -A\omega^2 \cos(\omega t + \phi)$
- Moments of Inertia of …
    - Uniform Hoop or thin cylindrical shell about its cylindrical axis: $I_{cm} = mR^2$
    - Uniform rigid rod about its center of mass: $I_{cm} = \frac{1}{12} mL^2$
    - Uniform Solid cylinder or disk about its cylindrical axis: $I_{cm} = \frac{1}{2} mR^2$
        - Okay, I didn't do this one during the review.
    - Use the parallel axis theorem to find the moment of inertia of any of these about any other axis.
    - A quick note about moments of inertia and the AP Exam. If you need the equation for a Moment of Inertia to solve a problem, it will be provided. And, while you do not need to memorize the equations for moments of inertia of various objects, you do need to be able to determine relative relationships between various moments of inertia of objects and those are just based on the basic equation for moment of inertia: $I = \sum_i m_i r_i^2$