Flipping Physics Lecture Notes:
LC Circuit Equation Derivations
http://www.flippingphysics.com/lc-circuit-derivations.html

We have already learned the basics of how an LC circuit works. Now let's derive equations for the LC Circuit, starting with the total energy in the circuit:

$$U_t = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{q^2}{2C} + \frac{1}{2}Li^2$$

- Typically, we use uppercase symbols for constants and lowercase symbols for variables.
- We know $I_{max} \rightarrow q = 0 \;\&\; Q_{max} \rightarrow i = 0$
- We can take the derivative with respect to time of the total energy equation. We know the derivative of total energy in the LC circuit equals zero because these are all ideal components with zero resistance. In other words, no energy is being dissipated from the system.

$$\Rightarrow \frac{dU_t}{dt} = \frac{d}{dt}\left(\frac{q^2}{2C} + \frac{1}{2}Li^2\right) = 0$$

- We need to use the chain rule for both energy expressions because time is not a variable in either energy expression, however, both charge, q, and current, i, are changing with respect to time.

$$\Rightarrow 0 = \frac{d}{dt}\left(\frac{q^2}{2C}\right) + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) \Rightarrow 0 = \left(\frac{2q}{2C}\right)\frac{dq}{dt} + \left(\frac{2Li}{2}\right)\frac{di}{dt}$$

$$\Rightarrow 0 = \left(\frac{q}{C}\right)\frac{dq}{dt} + (Li)\frac{di}{dt} \;\&\; i = \frac{dq}{dt} \;\&\; \frac{di}{dt} = \frac{d^2q}{dt^2}$$

$$\Rightarrow 0 = \left(\frac{q}{C}\right)i + (Li)\frac{d^2q}{dt^2} = \frac{q}{C} + (L)\frac{d^2q}{dt^2} \Rightarrow -\frac{q}{C} = (L)\frac{d^2q}{dt^2} \Rightarrow \frac{d^2q}{dt^2} = -\frac{1}{LC}q$$

- The equation definition for simple harmonic motion is: $\frac{d^2x}{dt^2} = -\omega^2 x$
- Therefore, we know the angular frequency of an LC circuit And we can determine the period of an LC Circuit:

$$\Rightarrow \omega_{LC}^2 = \frac{1}{LC} \Rightarrow \omega_{LC} = \frac{1}{\sqrt{LC}}$$

$$\& \;\omega = 2\pi f = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} \Rightarrow T_{LC} = \frac{2\pi}{1/\sqrt{LC}} \Rightarrow T_{LC} = 2\pi\sqrt{LC}$$

- And we know a general equation which satisfies the simple harmonic motion equation:

$$x(t) = A\cos(\omega t + \phi) \Rightarrow q(t) = Q_{max}\cos\left(\frac{t}{\sqrt{LC}} + \phi\right) \Rightarrow q(t) = Q_{max}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

- For this specific LC circuit the initial charge on the capacitor is Q_{max}, therefore, the phase constant is zero.

$$\& \; i = \frac{dq}{dt} \Rightarrow i(t) = \frac{d}{dt}\left[Q_{max}\cos\left(\frac{t}{\sqrt{LC}}\right)\right] = -Q_{max}\sin\left(\frac{t}{\sqrt{LC}}\right)\frac{d}{dt}\left(\frac{t}{\sqrt{LC}}\right)$$

- We can also determine current in an LC circuit as a function of time and an equation relating current maximum to charge maximum.

$$\Rightarrow i(t) = -\frac{Q_{max}}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right) \Rightarrow I_{max} = \frac{Q_{max}}{\sqrt{LC}} \Rightarrow i(t) = -I_{max} \sin\left(\frac{t}{\sqrt{LC}}\right)$$

- We can also derive the current maximum using the equation for total energy in the LC circuit.

$$U_t = U_C + U_L = \frac{q^2}{2C} + \frac{1}{2}Li^2 = \frac{Q_{max}^2}{2C} + 0 = 0 + \frac{1}{2}LI_{max}^2 \Rightarrow \frac{Q_{max}^2}{C} = LI_{max}^2$$

$$\Rightarrow I_{max}^2 = \frac{Q_{max}^2}{LC} \Rightarrow I_{max} = \frac{Q_{max}}{\sqrt{LC}}$$

- We can determine equations for energy as functions of time.

$$U_C = \frac{q^2}{2C} \Rightarrow U_C(t) = \frac{\left[Q_{max}\cos\left(\frac{t}{\sqrt{LC}}\right)\right]^2}{2C} \Rightarrow U_C(t) = \frac{Q_{max}^2}{2C}\cos^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$U_L = \frac{1}{2}Li^2 \Rightarrow U_L(t) = \frac{1}{2}L\left[-\frac{Q_{max}}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)\right]^2 = \left(\frac{1}{2}L\right)\left(\frac{Q_{max}^2}{LC}\right)\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$\Rightarrow U_L(t) = \frac{Q_{max}^2}{2C}\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$U_t(t) = U_C(t) + U_L(t) = \frac{Q_{max}^2}{2C}\cos^2\left(\frac{t}{\sqrt{LC}}\right) + \frac{Q_{max}^2}{2C}\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$\Rightarrow U_t(t) = \left(\frac{Q_{max}^2}{2C}\right)\left[\cos^2\left(\frac{t}{\sqrt{LC}}\right) + \sin^2\left(\frac{t}{\sqrt{LC}}\right)\right] \Rightarrow U_t(t) = \frac{Q_{max}^2}{2C}$$ & $\sin^2\theta + \cos^2\theta = 1$