Flipping Physics Lecture Notes:

AP Physics C: Rotational Dynamics Review – 2 of 2 (Mechanics)

- $\vec{\tau} = \vec{r} \times \vec{F} \neq \vec{F} \times \vec{r}$
  - Torque is the cross product (also called the vector product) of $\vec{r}$ & $\vec{F}$.
    - Torque is a vector!
  - $\vec{r}$ is the position vector from the axis of rotation to the location of the force, $\vec{F}$.
  - Magnitude of torque → $\tau = rF\sin\theta$
  - The order does matter! ($\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$)
  - Cross product is the area of the parallelogram with sides $\vec{r}$ & $\vec{F}$.
- In case you forgot how to do the cross product. Example: $\vec{A} = -\hat{i} + \hat{j} - 2\hat{k}$ & $\vec{B} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 2 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix}\hat{i} - \begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix}\hat{j} + \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix}\hat{k}$$

$$\Rightarrow \vec{A} \times \vec{B} = \left[(1)(4) - (-2)(-3)\right]\hat{i} - \left[(-1)(4) - (-2)(2)\right]\hat{j} + \left[(-1)(-3) - (1)(2)\right]\hat{k}$$

$$\Rightarrow \vec{A} \times \vec{B} = \left[4 - 6\right]\hat{i} - \left[-4 + 4\right]\hat{j} + \left[3 - 2\right]\hat{k} = \boxed{-2\hat{i} + \hat{k}}$$

- An object is in *Translational* Equilibrium if the net force acting on it equals zero, which means the object is not accelerating: $\sum \vec{F} = 0 = m\vec{a} \Rightarrow \vec{a} = 0$
- An object is in *Rotational* Equilibrium if the net torque acting on it equals zero, which means the object is not *angularly* accelerating: $\sum \vec{\tau} = 0 = I\vec{\alpha} \Rightarrow \vec{\alpha} = 0$ (must identify axis of rotation)
  - This means the object is either not rotating or has a constant angular velocity.
  - If an object is in translational equilibrium and in rotational equilibrium about *one* axis of rotation, then the object is in rotational equilibrium about *any* axis of rotation.
- $\vec{L}$ is Angular Momentum and it is a vector!
  - $\sum \vec{F} = m\vec{a} \Rightarrow \sum \vec{\tau} = I\vec{\alpha}$ & $\sum \vec{F} = \dfrac{d\vec{p}}{dt} \Rightarrow \sum \vec{\tau} = \dfrac{d\vec{L}}{dt}$
- For a *particle* or any object which is *not rotating*:
  - Just like torque, we have a cross product equation for angular momentum: $\vec{L} = \vec{r} \times \vec{p}$
    - r is the position vector from the axis of rotation to the location of the center of mass of the moving object.
    - And a magnitude equation for angular momentum: $L = rmv\sin\theta$
      - With this equation, need to use Right Hand Rule to find direction.
  - Yes, a particle or a rigid object which is not rotating can have an angular momentum!
- For a *rigid object with shape*: $\vec{L} = I\vec{\omega}$
  - Units for angular momentum: $\vec{L} = I\vec{\omega} \Rightarrow \left(kg \cdot m^2\right)\left(\dfrac{rad}{s}\right) = \dfrac{kg \cdot m^2 \cdot rad}{s} = \dfrac{kg \cdot m^2}{s}$
- Derivation of conservation of *linear* momentum: $\sum \vec{F}_{external} = \dfrac{d\vec{p}}{dt} = 0 \Rightarrow \sum \vec{p}_i = \sum \vec{p}_f$
- Derivation of conservation of *angular* momentum: $\sum \vec{\tau}_{external} = \dfrac{d\vec{L}}{dt} = 0 \Rightarrow \sum \vec{L}_i = \sum \vec{L}_f$

- Note the similarities between the two, please.
- Remember net torque requires the axis of rotation to be identified, which means the axis of rotation needs to be identified for conservation of angular momentum

- Conservation of Angular Momentum Example: A piece of gum with mass, m, and velocity, v, is spat at a solid cylinder of mass, M, radius, R, and moment of inertia $\frac{1}{2}MR^2$. The cylinder is on a horizontal axis through its center of mass and is initially at rest. The line of action of the gum is located horizontally a height, y, above the axis of the cylinder. If the gum sticks to the cylinder, what is the final angular velocity of the gum/cylinder system? The **Drawing!!**
  - Gum knowns: $m = m_g, v = v_{gi}$ Cylinder knowns: $M = m_c, \omega_{ic} = 0, R, I_c = \frac{1}{2}m_c R^2$
  - Solving for $\omega_f = ?$ (will be the same for both gum and cylinder)
  - Know angular momentum is conserved because: $\sum \vec{\tau}_{external} = \frac{d\vec{L}}{dt} = 0$
  - $\sum \vec{L}_i = \sum \vec{L}_f \Rightarrow \vec{L}_{gi} + \vec{L}_{ci} = \vec{L}_{gf} + \vec{L}_{cf} \Rightarrow r_{gi} m_g v_{gi} \sin\theta_{gi} + 0 = r_{gf} m_g v_{gf} \sin\theta_{gf} + I_c \omega_f$
  - $\sin\theta_{gi} = \frac{O}{H} = \frac{y}{r_{gi}} \Rightarrow y = r_{gi}\sin\theta_{gi} \Rightarrow y m_g v_{gi} = r_{gf} m_g v_{gf} \sin\theta_{gf} + I_c \omega_f$
  - $v_{gf} = v_t = R\omega_f \Rightarrow y m_g v_{gi} = R m_g R \omega_f \sin 90 + \frac{1}{2} m_c R^2 \omega_f$
  - $\Rightarrow y m_g v_{gi} = R^2 \omega_f \left(m_g + \frac{m_c}{2}\right) \Rightarrow \boxed{\omega_f = \frac{y m_g v_{gi}}{R^2\left(m_g + \frac{m_c}{2}\right)}}$

FYI: Sawdog, one of my Quality Control Team members, pointed out that, after colliding with the cylinder, the gum is moving in a circle, so it's angular momentum can be described using $I_g \omega_f$. More specifically: $\vec{L}_{gf} = I_g \omega_f = \left(m_g r_g^{\,2}\right)\omega_f = m_g R^2 \omega_f$ It's a slightly different solution that results in the same answer.