Flipping Physics Lecture Notes:

Work as the Dot Product
http://www.flippingphysics.com/work-dot-product.html

In our algebra-based physics, we defined work as: $W = Fd\cos\theta$
Where F is the force doing the work, d is the displacement of the object, and θ is the angle between those two vectors. We use the magnitudes of the force and the displacement in this equation.
Please remember the units for work are newtons times meters or joules:

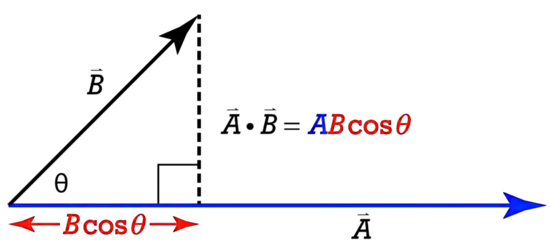
$W = Fd\cos\theta \Rightarrow W(units) = N \bullet m = joules$

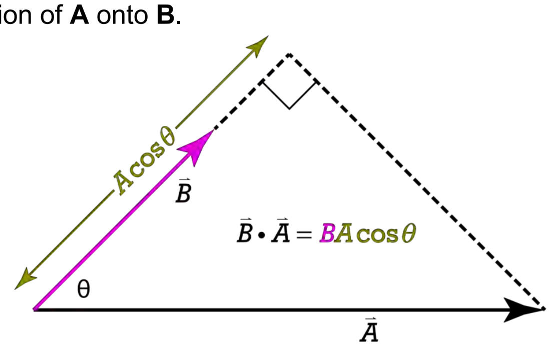
It is very important you realize this is the ***work done by a <u>constant force</u>***.
We will get to work done by a non-constant force later, not today.

In calculus-based physics we have the r position vector, so we define work done by a constant force as as:
$W = F\Delta r\cos\theta$
More generally, work done by a constant force is defined using the dot product (or scalar product) as:
$W = \vec{F} \bullet \Delta\vec{r}$
This means we get to review the dot product (or scalar product), Yea!
Recall that the dot product of A and B is this: $\vec{A} \bullet \vec{B} = AB\cos\theta$
So, it should be clear that the dot product of force and displacement equals work: $W = \vec{F} \bullet \Delta\vec{r} = F\Delta r\cos\theta$

I want to make sure you are aware that the scalar product is commutative. In other words, the order of the two vectors is irrelevant. $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A} \Rightarrow AB\cos\theta = BA\cos\theta$

The scalar product **A** · **B** is equal to the magnitude of **A** times Bcosθ.
And Bcosθ is the projection of **B** onto **A**. Which means the dot product of **A** and **B** is also equal to the magnitude of **B** times Acosθ, where Acosθ is the projection of **A** onto **B**.





$$\cos\theta = \frac{Adjacent}{Hypotenuse} = \frac{B_x}{B} \Rightarrow B_x = B\cos\theta$$

Let's do a simple unit vector example to recall how to do the dot product.

$\vec{A} = 4\hat{i} + 3\hat{j}$ & $\vec{B} = 2\hat{i} + 1\hat{j}$; $\vec{A} \bullet \vec{B} = ?$

$$(4+3)(2+1) = (4)(2) + (4)(1) + (3)(2) + (3)(1)$$

$$\vec{A} \bullet \vec{B} = \left[4\hat{i} + 3\hat{j}\right] \bullet \left[2\hat{i} + 1\hat{j}\right] = 4\hat{i} \bullet 2\hat{i} + 4\hat{i} \bullet 1\hat{j} + 3\hat{j} \bullet 2\hat{i} + 3\hat{j} \bullet 1\hat{j}$$

$$\Rightarrow \vec{A} \bullet \vec{B} = (4)(2)\cos(0) + (4)(1)\cos(90) + (3)(2)\cos(90) + (3)(1)\cos(0)$$

$$\Rightarrow \vec{A} \bullet \vec{B} = (4)(2) + (3)(1) = 11$$

In terms of physics, let's say we have an object which goes through a displacement $\Delta\vec{r} = \left[3.0\hat{i} + 1.0\hat{j}\right]m$ while experiencing a force $\vec{F} = \left[4.0\hat{i} + 2.0\hat{k}\right]N$. Then the work done on the object by the force is:

$$W = \vec{F} \bullet \Delta\vec{r} = \left[3\hat{i} + 1\hat{j} + 0\hat{k}\right] \bullet \left[4\hat{i} + 0\hat{j} + 2\hat{k}\right] = (3)(4) + (1)(0) + (0)(2) = 12J$$

Those are basically just a bunch of numbers, let's do a more real example.
A 6.9 N force is applied at 59° to the left of the -y axis to a shopping cart. The shopping cart is displaced 7.0 m to the left. If this force applied keeps the shopping cart moving at a constant velocity, what is the work done by each force on the shopping cart?

$$\vec{F}_a = 6.9N\ @59°\ left\ of\ -y\ axis;\ \Delta\vec{r} = \left[-7.0\hat{i}\right]m;\ W_{F_a} = ?$$

$$\vec{F}_a = \left[-F_{ax}\hat{i} - F_{ay}\hat{j}\right]N$$

$$\vec{F}_a = \left[-(6.9)\sin(59)\hat{i} - (6.9)\cos(59)\hat{j}\right]N$$

$$W_{F_a} = \vec{F}_a \bullet \Delta\vec{r} = \left[-(6.9)\sin(59)\hat{i} - (6.9)\cos(59)\hat{j}\right] \bullet \left[-7.0\hat{i}\right] = \left[-(6.9)\sin(59)\right]\left[-7\right] + \left[-(6.9)\cos(59)\right]\left[0\right]$$

$$\Rightarrow W_{F_a} = 41.401 N \bullet m \approx 41J$$

$$W_{F_g} = \vec{F}_g \bullet \Delta\vec{r} = \left[(-mg)\hat{j}\right] \bullet \left[-7.0\hat{i}\right] = 0$$

$$W_{F_g} = \vec{F}_g \bullet \Delta\vec{r} = \left[(-mg)\hat{j}\right] \bullet \left[-7.0\hat{i}\right] = (-mg)(-7)\cos(90) = 0$$

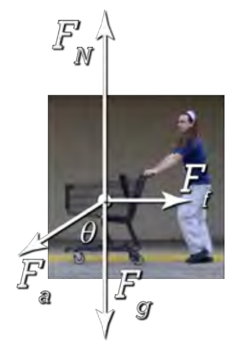
$$W_{F_g} = \vec{F}_g \bullet \Delta\vec{r} = \left[(-mg)\hat{j}\right] \bullet \left[-7.0\hat{i}\right] = \left[0\hat{i} + (-mg)\hat{j}\right] \bullet \left[-7.0\hat{i} + 0\hat{j}\right] = (0)(-7) + (-mg)(0) = 0$$

$$W_{F_N} = \vec{F}_N \bullet \Delta\vec{r} = \left[F_N\hat{j}\right] \bullet \left[-7.0\hat{i}\right] = 0$$

$$W_{F_f} = \vec{F}_f \bullet \Delta\vec{r} = \left[F_f\hat{i}\right] \bullet \left[-7.0\hat{i}\right] = -7F_f$$

$$\sum F_x = F_f - F_{a_x} = ma_x = m(0) = 0 \Rightarrow F_f = F_{a_x} = F_a \sin\theta$$

$$\Rightarrow W_{F_f} = (-7)(F_a \sin\theta) = (-7)(6.9)\sin(59) = -41.401 \approx -41J$$



In other words, all of the energy put into the cart by the force applied is dissipated by the force of friction.

Also notice that the net work done on the cart which is moving at a constant velocity equals zero. This is called foreshadowing.

$$W_{net} = W_{F_a} + W_{F_g} + W_{F_N} + W_{F_f} = 41.401 + 0 + 0 + (-41.401) = 0$$

$$W_{net} = 0\ (\text{at constant velocity})$$