Flipping Physics Lecture Notes:

Integral Introduction via Work
http://www.flippingphysics.com/integral-work.html

We have already worked with the work due to a constant force: $W = \vec{F} \bullet \Delta\vec{r} = F\Delta r\cos\theta$

Today we begin working with the work due to a non-constant force: $W = \int_{x_i}^{x_f} F_x\,dx$

- The work due to a non-constant force equals the integral, from position initial to position final, of force in the x-direction, with respect to x.

$\int \Rightarrow$ integral symbol

x_i & $x_f \Rightarrow$ initial and final conditions

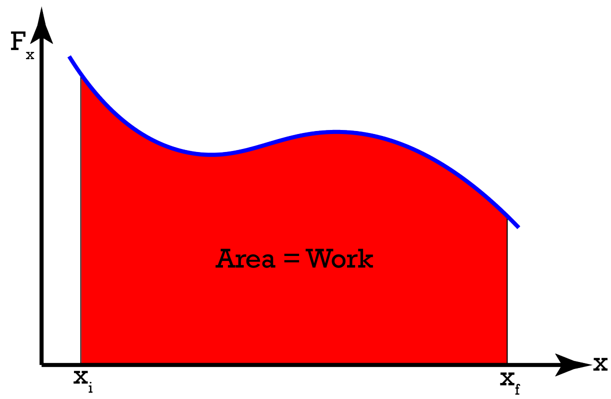
$F_x \Rightarrow$ function we are taking the integral of

$dx \Rightarrow$ taking the integral with respect to x

Due to the various levels of calculus familiarity, I do need to fully describe what the integral is and please realize, we are not going to concern ourselves with the difference between definite and indefinite integrals just yet. I know this may cause emotional distress for some of you. Please, breathe … and let it go.

A derivative is the rate of change of a function with respect to a variable, also known as the slope of a line.

The integral is the area "under" a function or a curve.
- The work is the area in the graph shown. →
- The quotes on "under" will be clarified later in this lesson.



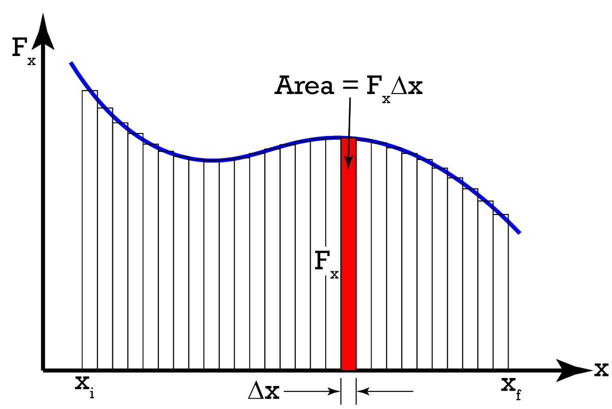
In order to determine the area "under" the curve, let's consider small changes in position or Δx. During those small changes in position the force can be considered to be approximately constant. And, because the force and displacement are in the same direction, the angle in the work equation is zero, and the work done by the force for this small change in position is:

$$W \approx \vec{F}_x \bullet \Delta\vec{x} \approx F_x\Delta x\cos\theta \approx F_x\Delta x\cos(0) \approx F_x\Delta x$$

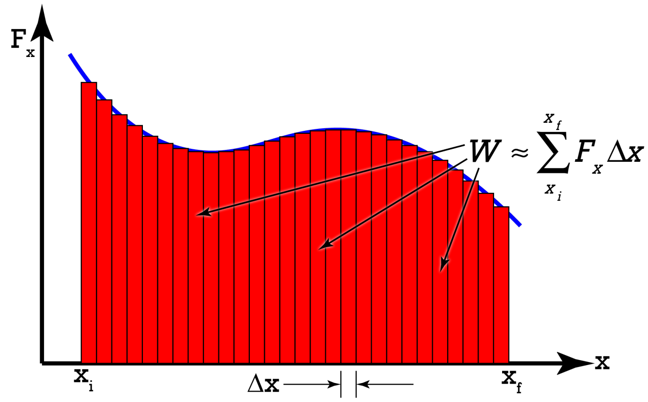
This is the area of the rectangle with sides F and Δx.

When we add all of the rectangles up for the positions from x_i to x_f we get:

$$W \approx \sum_{x_i}^{x_f} F_x\Delta x$$



Which is the approximate area "under" the function. In order to get the area instead of the *approximate* area, we take the limit as Δx gets infinitesimally small. And there will be an infinite number of area rectangles.



$$W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x$$

That is correct, we are adding up the areas of an infinite number of infinitesimally small rectangles and getting a real value. That is an integral.

$$W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F_x \Delta x = \int_{x_i}^{x_f} F_x \, dx = \text{Area "Under" Function}$$

Okay, that is all well and good, however, how do we *actually* do the math to determine an integral. Here we go. Realize an integral is also called an anti-derivative, because it is the reverse of a derivative. Starting with a simple derivative:

$$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$$

We can do the reverse to get the anti-derivative or integral: $y = \int 2x^1 \, dx = \frac{2x^2}{2} = x^2$

Notice I have added a simple step in the middle to follow the integral rule: $\int x^n \, dx = \left(\frac{1}{n+1}\right) x^{n+1}, n \neq -1$

Also, the 2 is constant and can be taken out from underneath the integral: $\int 2x \, dx = 2 \int x \, dx$

And we can do this again with a simple derivative and integral:

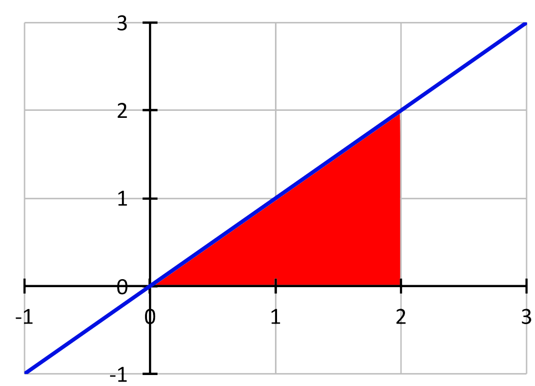
$$y = 3x^3 \Rightarrow \frac{dy}{dx} = 9x^2 \;\&\; y = \int 9x^2 \, dx = \frac{9x^3}{3} = 3x^3$$

Now we have advanced enough to be able to do integrals of functions without first doing the derivative:

$$y = \int 4x^4 \, dx = \frac{4x^5}{5} \;\&\; y = \int \left(6x^2 + x\right) dx = \int 6x^2 \, dx + \int x \, dx = \frac{6x^3}{3} + \frac{x^2}{2} = 2x^3 + \frac{x^2}{2}$$

Now let's show that the integral actually is the area "under" a function.

Let's start with determining the area "under" the function y = x from $x_i = 0$ to $x_f = 2$.



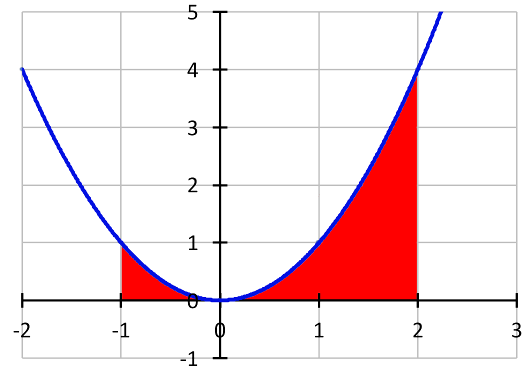
Clearly this is a triangle and has an area, between the line and the horizontal axis of:

$$A = \frac{1}{2} bh = \frac{1}{2}(2)(2) = 2$$

Determining same area using an integral:

$$A=\int_0^2 x\,dx=\left.\frac{x^2}{2}\right]_0^2=\left[\frac{x_f^{\,2}}{2}-\frac{x_i^{\,2}}{2}\right]_0^2=\frac{2^2}{2}-\frac{0^2}{2}=\frac{1}{2}(2)(2)=2$$

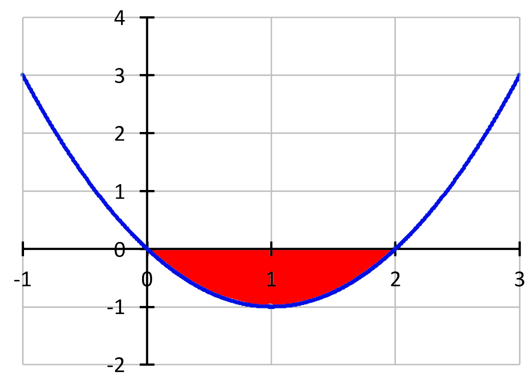
And you can see that the integral does give us the area of the triangle "under" this function.



Another example: Determine the area "under" the function $y = x^2$ from $x_i = -1$ to $x_f = 2$.

$$A=\int_{-1}^2 x^2\,dx=\left.\frac{x^3}{3}\right]_{-1}^2=\left[\frac{x_f^{\,3}}{3}-\frac{x_i^{\,3}}{3}\right]_{-1}^2=\frac{2^3}{3}-\frac{(-1)^3}{3}=\frac{8}{3}+\frac{1}{3}=3$$

And another example: Determine the area "under" the function $y = x^2 - 2x$ from $x_i = 0$ to $x_f = 2$.



$$A=\int_0^2\left(x^2-2x\right)dx=\left.\frac{x^3}{3}-\frac{2x^2}{2}\right]_0^2=\left.\frac{x^3}{3}-x^2\right]_0^2$$

$$\Rightarrow A=\left[\left(\frac{x_f^{\,3}}{3}-x_f^{\,2}\right)-\left(\frac{x_i^{\,3}}{3}-x_i^{\,2}\right)\right]_0^2$$

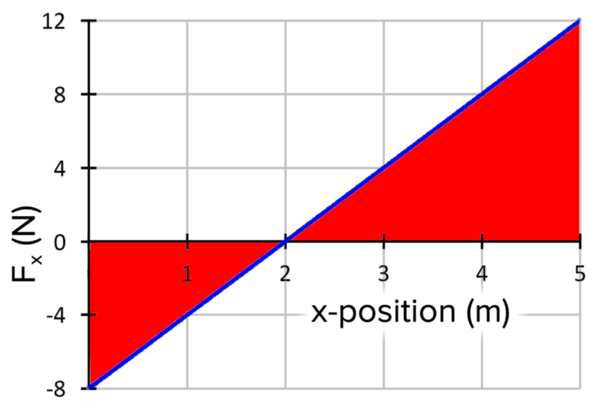
$$\Rightarrow A=\left(\frac{2^3}{3}-2^2\right)-\left(\frac{0^3}{3}-0^2\right)=\frac{8}{3}-4=-\frac{4}{3}$$

So, realize, by Area "Under" the Function we mean specifically:

- The area between the function and the x-axis.
- Area above the x-axis is positive.
- Area below the x-axis is negative.

And, to bring it back to where we started. Let's do one last example:

Determine the work done on an object by the force $F_x = (4x - 8)$ N as the object moves from $x_i = 0$ to $x_f = 5$ m.

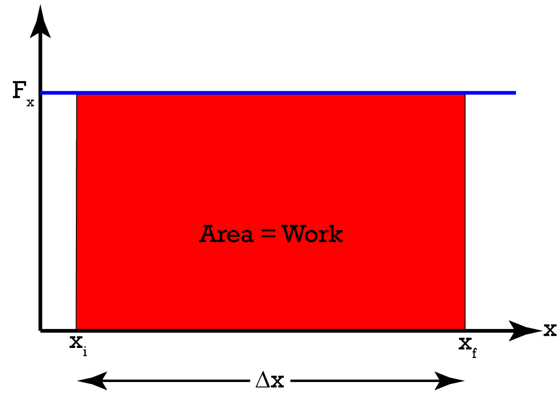


First off, we cannot use $W=\vec{F}\bullet\Delta\vec{r}=F\Delta r\cos\theta$ because the force is not constant.

So, we use $W=\int_{x_i}^{x_f}F_x\,dx$

$$W=\int_{x_i}^{x_f}F_x\,dx=\int_0^5(4x-8)\,dx=\left.\frac{4x^2}{2}-8x\right]_0^5=\left.2x^2-8x\right]_0^5=\left[(2)(5^2)-(8)(5)\right]-[0]=50-40=10J$$

Also, I keep referring to the integral work equation as the equation for the work done by a non-constant force. The reality is that you can also use this equation to find the work due to a constant force in the same direction as the displacement of the object. The integral just works out to give you the area of the rectangle with sides force in the x-direction and displacement. Typically, however, it is easier to simply think of the dot product work equation as the one to use with constant forces and the integral work equation as the one to use with non-constant forces.



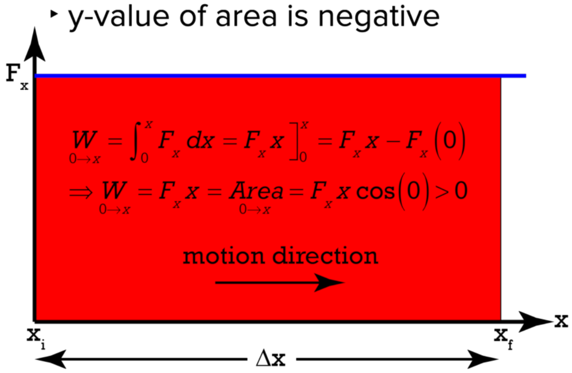
$$W = \int_{x_i}^{x_f} F_x \, dx = F_x x \Big]_{x_i}^{x_f} = F_x x_f - F_x x_i = F_x \left(x_f - x_i \right) = F_x \Delta x$$

$$W = \vec{F}_x \bullet \Delta\vec{x} = F_x \Delta x \cos\theta = F_x \Delta x \cos(0) = F_x \Delta x$$

And, of course, I need to clarify Area "Under" a Function a bit more:

When going in a ***positive*** direction:

- Area above the x-axis is ***positive***.
  - x-value of area is positive
  - y-value of area is positive
- Area below the x-axis is ***negative***.
  - x-value of area is positive
  - y-value of area is negative



When going in a ***negative*** direction:

- Area above the x-axis is ***negative***.
  - x-value of area is negative
  - y-value of area is positive
- Area below the x-axis is ***positive***.
  - x-value of area is negative
  - y-value of area is negative