Flipping Physics Lecture Notes:

Conservative Force and Potential Energy
http://www.flippingphysics.com/conservative-force-energy.html

In our previous video introducing conservative and nonconservative forces[1] we showed that the work done by a conservative force equals the change in potential energy associated with that force. With that we can include the integral definition of work:

$$W_{Conservative\ Force} = \int_{x_i}^{x_f} F_x\, dx = -\Delta U$$

Remembering that the integral means the sum of an infinite number of infinitesimally small rectangles with sides $F_x$ and dx, we can instead look at one of those infinitesimally small rectangles. Then we would also be looking at an infinitesimally small change in potential energy dU:

$$\Rightarrow F_x\, dx = -dU \Rightarrow F_x = -\frac{dU}{dx}$$

Which we can then rearrange to show that the component of a conservative force in the x-direction equals the negative of the derivative, with respect to x-position, of the potential energy associated with that force.

We can show the relationship is true using the spring force:

$$F_s = -\frac{dU_e}{dx} = -\frac{d}{dx}\left(\frac{1}{2}kx^2\right) = -\left(2\right)\left(\frac{1}{2}kx^1\right) = -kx$$

The spring force is negative because the spring force and the displacement from equilibrium position, x, are opposite in direction.

We can also show the relationship is true using the force of gravity in a constant gravitation field:

$$F_g = -\frac{dU_g}{dy} = -\frac{d}{dy}\left(mgy\right) = -mg$$

We take the derivative of potential energy with respect to y-position because the force of gravity is in the y-direction. And, the force of gravity is negative because the force of gravity is down.

Example: A conservative force $F_x = \left[-2.0x + 2.0x^2\right]\hat{i}\ N$ acts on an object as it moves from an x-position of 1.0 m to 2.0 m. What is the change in potential energy of the object? (You may assume this force is the only force acting on the object.)

First off, when you see the term “conservative force”, you need to immediately think of the equation Force in the x-direction equals the negative derivative with respect to x-position of the potential energy associated with that force. And realize this equation is not on the AP Physics C equation sheet. So, you need to memorize it.

$$F_x = -\frac{dU}{dx} \Rightarrow F_x\ dx = -dU \Rightarrow \int_{x_i}^{x_f} F_x\, dx = -\int_{U_i}^{U_f} dU = -\left[U\right]_{U_f}^{U_i} = -\left(U_f - U_i\right) \Rightarrow W_{F_x} = -\Delta U$$

Remember, we can rearrange any derivative to make an integral. We have arrived back at the equation we started with, just for good measure. And now we can solve for the change in potential energy:

$$\Rightarrow \Delta U = -\int_{x_i}^{x_f} F_x\, dx = -\int_1^2 \left(-2x + 2x^2\right) dx = -\left[-\frac{2x^2}{2} + \frac{2x^3}{3}\right]_1^2 = \left[x^2 - \frac{2x^3}{3}\right]_1^2$$

$$\Rightarrow \Delta U = \left(2\right)^2 - \frac{\left(2\right)\left(2\right)^3}{3} - \left(\left(1\right)^2 - \frac{\left(2\right)\left(1\right)^3}{3}\right) = \left(-\frac{4}{3}\right) - \left(\frac{1}{3}\right) = -\frac{5}{3} \approx -1.7J$$

[1] Conservative and Nonconservative Forces - http://www.flippingphysics.com/conservative-nonconservative.html

The change in potential energy associated with this conservative force is negative. While we do not know much about what is physically going on in this example, because we do not know much about this conservative force which I completely made up, we do know that the potential energy of the object has decreased in the direction of the motion of the object. What we can do is make a comparison to gravitational potential energy and the force of gravity. In that gravitational analogy, the object has moved down. The force of gravity and displacement of the object are in the same direction, so the work done on the object by the force of gravity is positive, that means the change in gravitational potential energy of the object is negative, so the gravitation potential energy of the object has decreased.

Gravitational Analogy:
- Force of gravity is down
- Displacement is down
- $W_{F_g} > 0 \rightarrow \Delta U_g < 0$
- $U_g$ has decreased

If you look at the graph of the force in our example, you can see the conservative force in our example is always in the positive x-direction between 1 and 2 meters. Because the displacement of the object is also in the positive x-direction, the work done by the force, or the area "under" the function, is positive. We know the change in potential energy associated with this conservative force equals the negative of that work, which is why the change in potential energy of the object is negative; the potential energy of the object has decreased.