Flipping Physics Lecture Notes:

Indefinite Integral Introduction and 4 Kinematic (UAM) Equation Derivations

http://www.flippingphysics.com/kinematic-derivation.html

Up to this point we have only worked with definite integrals. Today we introduce the indefinite integral. A definite integral has defined limits. An indefinite integral does not have defined limits.

Let's start with the derivative definition of acceleration to introduce indefinite integrals:

$$a=\frac{dv}{dt}\Rightarrow\int dv=\int a\,dt$$

Any derivative can be rearranged to form an antiderivative or an integral, however, notice we do not have limits on the integral. Integrating both sides gives us:

$$\Rightarrow v\left(t\right)=at+C$$ Please realize this integral assumes the acceleration is constant.

Whenever we take an integral, we need to add a constant of integration or C. What is C? Let's determine the value of the velocity function at the initial time or time equals zero seconds:

$$v\left(0\right)=a\left(0\right)+C\Rightarrow v_i=C$$

Notice that C is the initial condition. In this case the constant of integration is the initial velocity. In other words:

$$\Rightarrow v\left(t\right)=at+v_i\Rightarrow v_f=v_i+at$$

Note: v_i at $t_i=0$ & v_f at $t_f=t$

We have derived one of the Uniformly Accelerated Motion (or kinematic) equations!

Let's take a step back and point out that we only need to add one constant of integration C. If we added a constant of integration to both integrals:

$$\int dv=\int a\,dt\Rightarrow v\left(t\right)+C_1=at+C_2\Rightarrow v\left(t\right)=at+C_2-C_1=at+C$$

We just end up with C_2 minus C_1 which is still a constant which we can just call C!

Alright, back to the equation we derived. Again, any derivative can be rearranged to form an integral:

$$v_f=v_i+at\Rightarrow\frac{dx}{dt}=at+v_i\Rightarrow\int dx=\int\left(at+v_i\right)dt\Rightarrow x\left(t\right)=\frac{at^2}{2}+v_it+C$$

Again, we can solve for C by determining the value of the function at time equals zero seconds:

$$\Rightarrow x\left(0\right)=\frac{a\left(0\right)^2}{2}+v_i\left(0\right)+C\Rightarrow x_i=C\Rightarrow x\left(t\right)=\frac{at^2}{2}+v_it+x_i\Rightarrow x_f=x_i+v_it+\frac{1}{2}at^2$$

We have derived another one of the Kinematic (or Uniformly Accelerated Motion) equations!

So, notice the constant of integration, C, is the value of the function when time equals zero.

But wait, there's more! dv divided by dt is the same as dv over dx times dx over dt![1]

$$a=\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=\frac{dv}{dx}v\Rightarrow\int_{x_i}^{x_f}a\,dx=\int_{v_i}^{v_f}v\,dv\Rightarrow ax\Big]_{x_i}^{x_f}=ax_f-ax_i=a\Delta x=\frac{v^2}{2}\Big]_{v_i}^{v_f}=\frac{{v_f}^2}{2}-\frac{{v_i}^2}{2}$$

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[1] This is the Chain Rule: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

$$\Rightarrow v_f^{\,2} - v_i^{\,2} = 2a\Delta x \Rightarrow v_f^{\,2} = v_i^{\,2} + 2a\Delta x$$

We used a definite integral this time and we have another UAM equation!

And yes, there is one more kinematic equation we can solve for. We can solve for acceleration using one of the derived equations and substitute that into another derived equation:

$$v_f = v_i + at \Rightarrow a = \frac{v_f - v_i}{t} \;\&\; x_f = x_i + v_i t + \frac{1}{2}at^2 \Rightarrow x_f - x_i = v_i t + \frac{1}{2}\left(\frac{v_f - v_i}{t}\right)t^2$$

$$\Rightarrow \Delta x = v_i t + \frac{1}{2}\left(v_f - v_i\right)t = t\left(v_i + \frac{v_f}{2} - \frac{v_i}{2}\right) = t\left(\frac{v_i}{2} + \frac{v_f}{2}\right) \Rightarrow \Delta x = \frac{1}{2}\left(v_i + v_f\right)t$$

And yes, we have solved for the fourth, and oft ignored, kinematic equation.

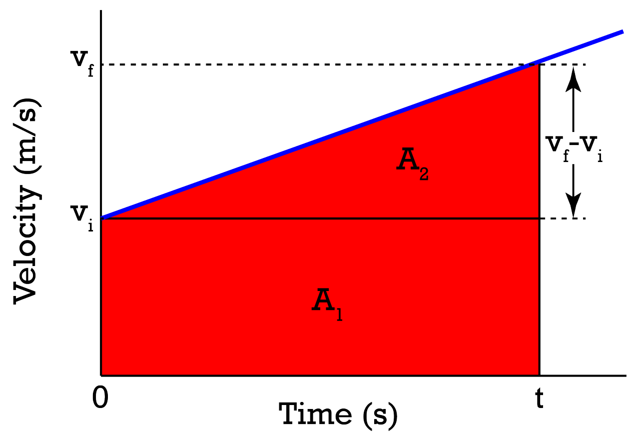
But if you would prefer to use an integral…
Remember an integral represents the Area "Under" the Function. Uniformly Accelerated Motion means the acceleration is constant. That function is a straight line on a velocity as a function of time graph.

$$\Delta x = \int_{t_i}^{t_f} v\,dt = \text{Area "under" Function} = A_1 + A_2 = h_1 b_1 + \frac{1}{2}h_2 b_2 = v_i t + \frac{1}{2}\left(v_f - v_i\right)t$$

And that is the same equation we had before when deriving the last UAM equation!

We could also have used the area of a trapezoid equation:

$$\Delta x = \int_{t_i}^{t_f} v\,dt = \frac{1}{2}\left(b_1 + b_2\right)h = \frac{1}{2}\left(v_i + v_f\right)t$$



Alternate solution for the third UAM equation using an indefinite integral:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v \Rightarrow \int a\,dx = \int v\,dv \Rightarrow \frac{v^2}{2}\left(x\right) = ax + C \Rightarrow \frac{v^2}{2}\left(x = 0\right) = a\left(0\right) + C$$

Notice the equation is for the velocity squared over 2 as a function of position. So, we substitute in an initial condition of initial position equals zero. We know that, at the initial position, the velocity is the initial velocity. And we can solve for the velocity final squared.

$$\Rightarrow \frac{v_i^{\,2}}{2} = C \Rightarrow \frac{v^2}{2}\left(x\right) = ax + \frac{v_i^{\,2}}{2} \Rightarrow v_f^{\,2} = v_i^{\,2} + 2ax$$

Realize that we do not know the initial position is zero, therefore, we need to substitute delta x in for x:

$$\Delta x = x_f - x_i = x - 0 = x$$

And we end with the same UAM equation we got using definite integrals.

$$v_f^{\,2} = v_i^{\,2} + 2ax \Rightarrow v_f^{\,2} = v_i^{\,2} + 2a\Delta x$$

I think it was easier to solve using definite integrals!