Flipping Physics Lecture Notes:

Center of Mass by Integration

http://www.flippingphysics.com/center-mass-integral.html

Previously we have determined the center of mass of a system of particles:[1]

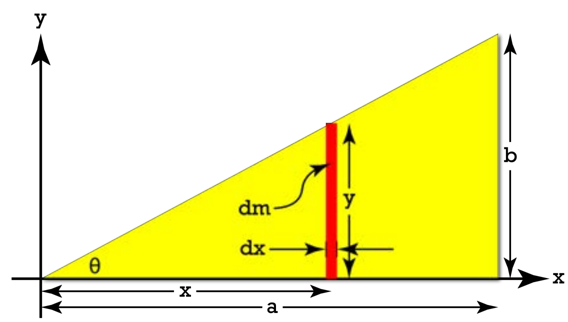
$$x_{cm}=\frac{m_1x_1+m_2x_2+\ldots}{m_1+m_2+\ldots}=\frac{\sum_i m_ix_i}{\sum_i m_i}=\frac{\sum_i m_ix_i}{m_{total}}$$

Where i goes from 1 to the total number of particles in the system.

Now we are going to find the center of mass of a rigid object with shape which is just an object made up of particles. Each of those particles will have a position x and a mass Δm: $x_{cm}=\frac{\sum_i x_i\Delta m_i}{m_{total}}$

However, realize we are going to consider the rigid object with shape to be made up of an infinite number of particles, which means we are going to be taking the limit as the number of particles goes to infinity and Δm becomes infinitesimally small and we call it dm. This is, of course, an integral:

$$x_{cm}=\lim_{\Delta m_i\to 0}\frac{\sum_i^{\infty} x_i\Delta m_i}{m_{total}}=\frac{1}{m_{total}}\int x\,dm$$



To understand how this works in practice, we are going to determine the x-position center of mass of a triangle of sides a and b and thickness t and uniform density ρ.

We start by breaking this triangle into an infinite number of pieces dm which each have a shape of a rectangular box with sides dx, y, and t. Each of these infinitesimally small rectangular boxes dm is located a distance x from the origin. But notice the issue here is that we are trying to take the integral of x with respect to mass. In order to do this, we need to find a relationship between x-position and mass. For this we use volumetric mass density:

$$\rho=\frac{m}{V}=\frac{m_t}{V_t}=\frac{dm}{dV}\Rightarrow dm=\rho dV$$

Both the triangle and the infinitesimally small pieces of the triangle dm have the same volumetric mass density. The density of the triangle equals the total mass of the triangle divided by the total volume of the triangle. The density of the infinitesimally small pieces have mass dm and volume dV.

We can determine the volume of dV. Remember it is a rectangular box of thickness t: $dV=yt\,dx$

And we can substitute that back into our equation for dm: $\Rightarrow dm=\rho dV=\rho\left(yt\,dx\right)$

Now let's look at the density[2] of the entire triangle: $\rho=\frac{m_t}{V_t}=\frac{m_t}{\frac{1}{2}abt}$

[1] https://www.flippingphysics.com/center-of-mass-particles.html

[2] Please realize the rigid object with shape is a triangle, however, the infinitesimally small pieces of that triangle dm are rectangular boxes. The infinitesimally small pieces of the larger object and the larger object do not have the same shape!

We can now that into our equation for dm: $$\Rightarrow dm = \rho\left(yt\,dx\right) = \left(\frac{2m_t}{abt}\right)\left(yt\right)dx = \left(\frac{2m_t y}{ab}\right)dx$$

Which we can substitute back into our equation for x_{cm}: $$\Rightarrow x_{cm} = \frac{1}{m_t}\int x\left(\frac{2m_t y}{ab}\right)dx$$

And we can take everything out from the integral and total mass cancels out: $$\Rightarrow x_{cm} = \frac{2}{ab}\int xy\,dx$$

Notice y is not constant and does change depending on the x location of dm. Therefore, in order to take this integral, we need to find a way to relate y and x. For this we turn to trigonometry!

$$\tan\theta = \frac{O}{A} = \frac{y}{x} = \frac{b}{a} \Rightarrow y = \frac{b}{a}x$$

Or you could use the slope intercept form a straight line to get the same relationship:

$$y = mx + y_0 \Rightarrow y = \frac{b}{a}x$$

And we can substitute that back into our x_{cm} equation: $$\Rightarrow x_{cm} = \frac{2}{ab}\int x\left(\frac{b}{a}x\right)dx = \frac{2}{a^2}\int x^2\,dx$$

Now we need limits for x. Notice x varies from 0 to a:

$$\Rightarrow x_{cm} = \frac{2}{a^2}\int_0^a x^2\,dx = \frac{2}{a^2}\left[\frac{x^3}{3}\right]_0^a = \frac{2}{a^2}\left[\frac{a^3}{3} - \frac{0^3}{3}\right] = \frac{2}{3}a$$

So, the x center of mass of a triangle is two-thirds the length of the triangle from the tip of the triangle. Realize that the height of the triangle, b, is irrelevant as far as to where the x center of mass is located. And, objects in projecile motion rotate around their center of mass…