Flipping Physics Lecture Notes:

Nonuniform Density Center of Mass
http://www.flippingphysics.com/center-mass-nonuniform.html

Previously we learned how to determine the center of mass of uniform, rigid objects with shape.[1] Today we use the same approach to determine the center of mass of a *nonuniform*, rigid object with shape. The easiest way to do this is by way of example.

But first, realize there are 3 different mass densities, each represented by a lowercase, Greek letter:

$$(rho)\ \rho = \frac{m}{V}\ (\text{volumetric mass density})$$

$$(sigma)\ \sigma = \frac{m}{A}\ (\text{surface mass density})$$

$$(lambda)\ \lambda = \frac{m}{L}\ (\text{linear mass density})$$

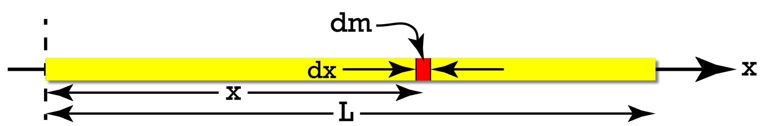
Example: Determine the x-position center of mass of a horizontally oriented rod with a length of 0.65 m and $\lambda = \left[43 - 21x^2\right]\frac{g}{m}$.

Knowns: $L = 0.65m;\ \lambda = \left[43 - 21x^2\right]\frac{g}{m};\ x_{cm} = ?$

Clearly this problem deals with linear mass density, λ. And we are trying to find the center of mass of a rigid object with shape. That means we need to use the equation: $x_{cm} = \frac{1}{m_t}\int x\,dm$

Clearly, in order to use this equation, we need to determine the mass of the rod. For this we use an integral:

$$m_t = \int dm$$



However, we need to relate dm to λ:

$$\lambda = \frac{dm}{dx} \Rightarrow dm = \lambda\,dx$$

Which we can substitute back into the equation:

$$m_t = \int dm = \int \lambda\,dx = \int_0^{0.65}\left(43 - 21x^2\right)dx = \left[43x - \frac{21x^3}{3}\right]_0^{0.65} = \left[43x - 7x^3\right]_0^{0.65}$$

Notice the limits for x go from 0 to L or 0 to 0.65 m.

$$\Rightarrow m = \left[(43)(0.65) - (7)(0.65)^3\right] - (0) = 26.0276g$$

It is important to realize $\lambda \neq \frac{m_t}{L}$ instead $\lambda_{avg} = \frac{m_t}{L}$.

And now we do something very similar with the x-position center of mass equation:

$$x_{cm} = \frac{1}{m_t}\int x\,dm = \frac{1}{m_t}\int x\lambda\,dx = \frac{1}{m_t}\int_0^{0.65}(x)\left(43 - 21x^2\right)dx = \frac{1}{m_t}\int_0^{0.65}\left(43x - 21x^3\right)dx = \frac{1}{m_t}\left[\frac{43x^2}{2} - \frac{21x^4}{4}\right]_0^{0.65}$$

$$\Rightarrow x_{cm} = \left(\frac{1}{26.0276}\right)\left(\frac{(43)(0.65)^2}{2} - \frac{(21)(0.65)^4}{4}\right) - 0 = 0.312998m \approx 31cm$$

[1] http://www.flippingphysics.com/center-mass-integral.html

The center of mass of a uniform rod would be at its geometric center or 32.5 cm from its left end. Therefore this result makes sense because we would expect this object which slowly decreases in density from left to right, to have a center of mass slightly to the left of 32.5 cm.