Flipping Physics Lecture Notes:

Centripetal Acceleration Derivation

http://www.flippingphysics.com/centripetal-acceleration-derivation.html

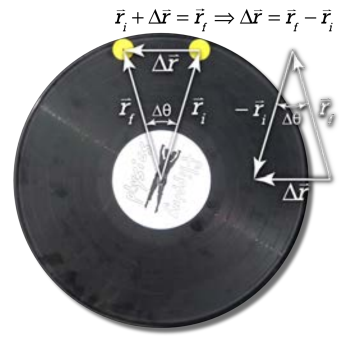
We have already been introduced to centripetal acceleration[1] and centripetal force[2], however, today we are going to derive the equation and the direction for centripetal acceleration!

$$a_c = \frac{v_t^2}{r} = r\omega^2 \text{ \& } \sum \vec{F}_{in} = m\vec{a}_c$$

In order to do that, we start by reviewing the basic equation for average linear acceleration: $\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t}$

Now let's look at an object moving at a constant angular velocity. Because the object has a constant angular velocity, the angular acceleration of the object equals zero:

$$\vec{\omega} = \text{constant} \Rightarrow \vec{\alpha} = \frac{\Delta \vec{\omega}}{\Delta t} = \frac{0}{\Delta t} = 0$$



After the object has gone through a small change in angular position, Δθ, we can identify several things about this motion. First the initial and final positions, $\vec{r}_i$ & $\vec{r}_f$, and then the displacement of the object $\Delta\vec{r}$.

Now, it is subtle, but important, you realize what is shown here via tip-to-tail vector addition is actually $\vec{r}_i + \Delta\vec{r} = \vec{r}_f$ but that is the same as $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$.

What we have created here is an isosceles triangle with an angle Δθ, two sides of equal magnitude, $\vec{r}_i$ & $\vec{r}_f$, and a third side $\Delta\vec{r}$. We know $\vec{r}_i$ & $\vec{r}_f$ are equal in magnitude because the radius of the circle being described by the object is constant. In other words, $\left\|\vec{r}_i\right\| = \left\|\vec{r}_f\right\| = r$, where r is the radius of the circle.

Now let's add the velocity of the object. Because the object is moving in a circle, the linear velocity of the object is its tangential velocity, is always tangent to the circle, and is therefore always at a 90-degree angle with the radius.

[3]We can add the initial and final velocities, $\vec{v}_i$ & $\vec{v}_f$, which correspond to the initial and final positions, $\vec{r}_i$ & $\vec{r}_f$, when the object goes through the same small change in angular position, Δθ. Because these velocities are both at right angles to $\vec{r}_i$ & $\vec{r}_f$, we know that the angle between $\vec{v}_i$ & $\vec{v}_f$ is also Δθ and we can create a similar isosceles triangle using $\Delta\vec{v} = \vec{v}_f - \vec{v}_i \Rightarrow \vec{v}_i + \Delta\vec{v} = \vec{v}_f$.

Notice the direction of the change in velocity of the object is directed inward towards the center of the circle. In other words, when an object is moving at a constant angular velocity, the change in linear velocity of the object is always directed towards the center of the circle. This is the direction of the centripetal acceleration of the object. It is always in, towards the center of the circle!

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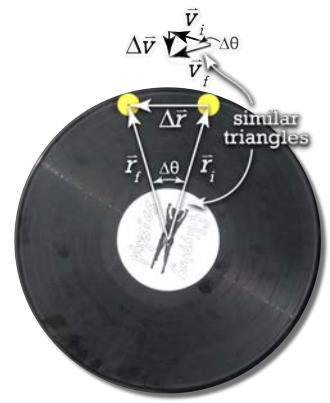
[1] https://www.flippingphysics.com/centripetal-acceleration.html
[2] https://www.flippingphysics.com/centripetal-force.html
[3] Please ignore the black vs. white colors in this graphic. I'll fix them when I make the video.

Next, realize, because the object is moving at a constant angular velocity, the magnitude of the tangential velocity of the object is also constant and we can define that velocity as v_t. $\left\|\vec{v}_i\right\| = \left\|\vec{v}_f\right\| = v_t$

Because both of these isosceles triangles have the same angle Δθ, they are similar triangles. (One condition for similar triangles is that two pairs of corresponding sides are proportional and the corresponding angles between them are equal.) Because these are similar triangles we know:

$$\frac{\Delta v}{v_t} = \frac{\Delta r}{r} \Rightarrow \Delta v = \frac{(\Delta r)(v_t)}{r}$$

I have removed the vector symbols because we are only comparing the lengths of the sides of these triangles and have rearranged the equation to solve for change in velocity.



And now we can go back to the original equation for linear acceleration and substitute in for change in velocity, Δv.

$$a = \frac{\Delta v}{\Delta t} = \left(\frac{1}{\Delta t}\right)\left(\frac{(\Delta r)(v_t)}{r}\right) = \left(\frac{v_t}{r}\right)\left(\frac{\Delta r}{\Delta t}\right)$$

And we can take the limit of this equation as Δt decreases to an infinitesimally small change in time.

$$\lim_{\Delta t \to 0}\left(\frac{\Delta r}{\Delta t}\right) = \frac{dr}{dt} = v_t$$

This is the definition of a derivative. Which means we get the equation for the acceleration in the in-direction, or centripetal acceleration to be:

$$\Rightarrow a = \left(\frac{v_t}{r}\right)(v_t) \Rightarrow a_c = \frac{v_t^{\,2}}{r}$$

Again, for an object moving at a constant angular velocity, the *magnitude* of the tangential velocity remains constant, however, the *direction* of the tangential velocity is changing, and it is the change in the direction of the tangential velocity as a function of time which is the acceleration in the in-direction, and it is called centripetal acceleration.