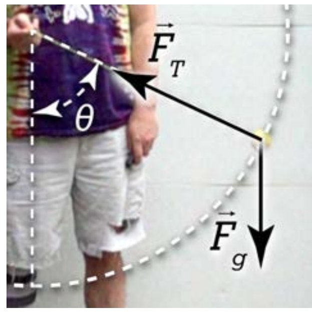
Flipping Physics Lecture Notes:

Nonuniform Circular Motion - Force of Tension in a Rope

http://www.flippingphysics.com/non-uniform-circular-motion-tension.html

Example: A ball of mass *m* on the end of a string of length *L* moves in a vertical circle with a non-constant angular speed *ω*. The string forms an angle *θ* with the vertical as shown. Determine the tension in the rope in terms of *m*, *L*, *ω, θ*, and known constants.



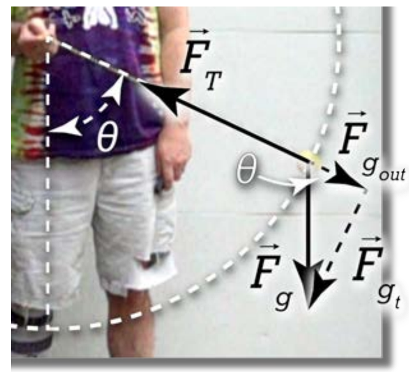
Knowns: *m*, *L*, *ω, θ* → F_T = ?

We start by drawing the Free Body Diagram.

Then we break forces into components in the in direction and the tangential direction.

Notice the angle *θ* we use to determine the components of the force of gravity is the same angle *θ* defined in the problem.

$$\sin\theta = \frac{O}{H} = \frac{F_{g_t}}{F_g} \Rightarrow F_{g_t} = mg\sin\theta \quad \& \quad \cos\theta = \frac{A}{H} = \frac{F_{g_{out}}}{F_g} \Rightarrow F_{g_{out}} = mg\cos\theta$$



Redraw the Free Body Diagram and sum the forces in the tangential direction.

$$\sum F_t = F_{g_t} = ma_t \Rightarrow mg\sin\theta = ma_t \Rightarrow a_t = g\sin\theta$$

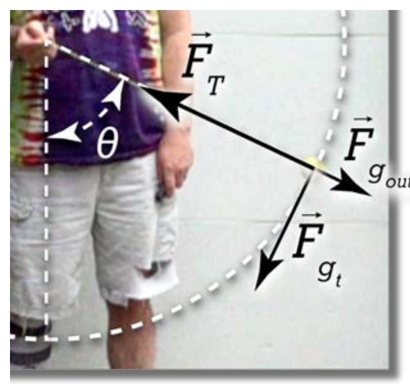
We now know the value of the tangential acceleration of the ball, however, this does not help us solve for the force of tension in the string. ☺

Sum the forces in the in-direction, remembering that inward is positive and outward is negative.

$$\sum F_{in} = F_T - F_{g_{out}} = ma_c \Rightarrow F_T = F_{g_{out}} + ma_c = mg\cos\theta + mr\omega^2$$

$$\Rightarrow F_T = mg\cos\theta + mL\omega^2$$

This answer matches what we previously showed visually.[1]



But notice we can also solve for the minimum angular speed to keep the ball moving in a circle. When the ball is at the top of the circle, at that minimum speed the force of tension is reduced down to zero and the angle is 180°.

$$\Rightarrow 0 = mg\cos(180) + mL\omega^2 \Rightarrow -mg\cos(-1) = mL\omega^2 \Rightarrow g = L\omega^2 \Rightarrow \omega_{min} = \sqrt{\frac{g}{L}}$$

Which is the same answer we got for the "Minimum angular speed necessary to keep water in a vertically revolving bucket"![2]

Please realize the angular speed of the ball decreases as it moves up toward the top of the circle, and increases as it moves down away from the top of the circle. Therefore, this minimum angular speed we are talking about is only at the singular point which is at the very top of the path, when the string is completely vertical. Before and after the ball is at the top of the path the angular speed of the ball will be larger than this minimum angular speed.

[1] http://www.flippingphysics.com/non-uniform-circular-motion-ball.html
[2] https://www.flippingphysics.com/water-bucket-minimum-speed.html