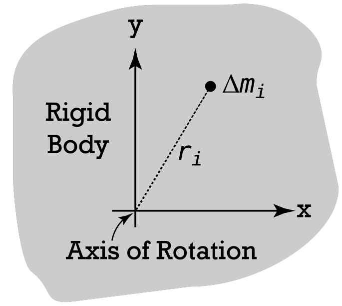
Flipping Physics Lecture Notes: Uniform Thin Hoop Rotational Inertia Derivation
http://www.flippingphysics.com/rotational-inertia-thin-hoop.html

We have already[1] derived the equation for the Moment of Inertia or Rotational Inertia of a system of particles: $I = \sum_i m_i (r_i)^2$

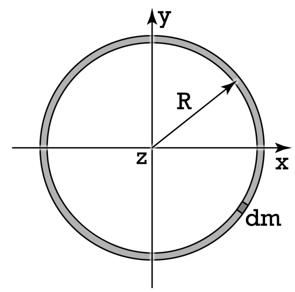


Now we are going to determine the Moment of Inertia or Rotational Inertia of Rigid Objects with Shape. A Rigid Body is simply an object that does not change shape easily and is larger than a point particle. We are going to treat that Rigid Body as a large number of small particles each with a mass Δm_i located a distance r from the axis of rotation.

Then we take the limit of that to increase the number of particles to an infinite number of infinitesimally small particles. The Rotational Inertia of the object will then be given by the equation:

$$I = \lim_{\Delta m_i \to 0} \sum_i \Delta m_i (r_i)^2$$

This is, of course, the definition of an integral and each infinitesimally small particle Δm_i is called dm: $\Rightarrow I = \int r^2 \, dm$ (rigid body)



Now let's determine the Rotational Inertia of a Uniform Thin Hoop with a mass of M and a radius R about an axis perpendicular to the plane of the hoop which passes through its center. Let's call this the "z" axis.

- "Uniform" means the density of the hoop is constant.
- "Thin" means the thickness of the hoop is small enough relative to the radius of the hoop such that the thickness is negligible.
  - This means the distance from the axis of rotation to the *inside* of the uniform thin hoop is R.
  - And the distance from the axis of rotation to the *outside* of the uniform thin hoop is also R.
- Notice every particle dm is located the same distance R from the axis of rotation. This means r = R and we can take R out from under the integral.
- The integral of just dm is M.

$$I_z = \int r^2 \, dm = R^2 \int dm = R^2 M \Rightarrow I_z = MR^2$$

This Moment of Inertia makes sense because every particle dm is located a distance R from the axis of rotation, so all the mass is located a distance R from the axis of rotation, so the Rotational Inertia of a Uniform Thin Hoop is the same as the Rotational Inertia of a single particle with a total mass M located a distance R from the axis of rotation.

Notice that the length of the uniform thin ring in the z direction is not in these calculations. In other words, the moment of inertia of a uniform thin cylinder is also the mass of the cylinder times the radius of the cylinder squared. This again makes sense because all of the mass of the cylinder is still located a distance of the radius of the cylinder away from the axis of rotation.

Also notice a subtle but important difference between the use of lowercase and uppercase letters here. The lowercase r and dm refer to variables. r is a variable distance between the axis of rotation and the location of dm. dm is a mass which could be located anywhere on the hoop. Whereas uppercase R and M refer to constants. R is the constant radius of the hoop. M is the total, constant mass of the hoop. This idea of lowercase letters referring to variables and uppercase letters referring to constants is one which we will continue to see in physics.

Please do not confuse the equation for center of mass with the equation for rotational inertia!! $\vec{r}_{cm} = \frac{1}{m_{total}} \int \vec{r} \, dm$

[1] https://www.flippingphysics.com/moment-of-inertia.html