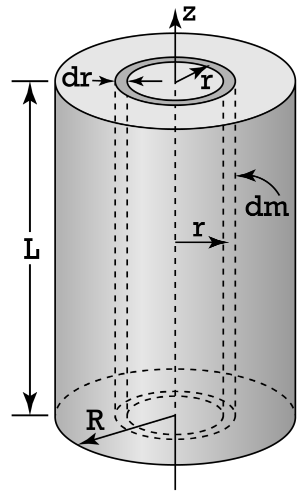
Flipping Physics Lecture Notes:

Uniform Solid Cylinder Moment of Inertia Derivation

http://www.flippingphysics.com/rotational-inertia-solid-cylinder.html



We have already derived the equations for the moment of inertia or rotational inertia of a Uniform Thin Hoop[1] and a Uniform Thin Rod[2]. Today we are deriving the moment of inertia of a uniform solid cylinder about its cylindrical axis.

The general equation for the rotational inertia of a rigid body is: $I = \int r^2 \, dm$

The specific question is: Determine the rotational inertia of a Uniform Solid Cylinder about its cylindrical axis in terms of its mass M, length L, and radius R.

First let's make sure we understand that M is the total mass of the solid cylinder, L is the total length of the solid cylinder, and R is the radius of the solid cylinder. This may seem obvious right now, however, there are going to be more, similar variables soon. Also, we need to review that "uniform" means that the cylinder is of uniform volumetric mass density, ρ.

Now we identify the infinitesimally small pieces, dm, of our uniform solid cylinder. They are in the shape of an extruded circle, in other words, each dm is in the shape of an infinitesimally thin toilet paper roll. The length of dm is also L, the radius of dm is r which varies from 0 to R, and the thickness of dm is an infinitesimally small thickness dr. Recall that because the thickness of dm is infinitesimally small, the radius r of dm goes to both the inner and outer edges of dm. Also, the mass of dm is … dm.

And remember the lowercase variable r refers to the variable radius of dm.
Whereas the uppercase letter R refers to the constant radius of the uniform solid cylinder.

Again, we have this issue where we are taking the integral of a linear variable r with respect to mass. What we need to do is get dm in terms of dr so we can take this integral. To do that, we start with the general equation for volumetric mass density and define the volume of dm as dV:

$$\rho = \frac{m}{V} = \frac{dm}{dV} \Rightarrow dm = \rho dV$$

Now we need to determine the volume of dV and substitute that back into the equation for dm:

$$dV = (2\pi r) L dr \Rightarrow dm = \rho 2\pi r L dr$$

Which we can now substitute into the equation for moment of inertia:

$$I_z = \int r^2 \rho 2\pi r L dr = \rho 2\pi L \int_0^R r^3 dr = \rho 2\pi L \left[\frac{r^4}{4}\right]_0^R = \rho 2\pi L \left[\frac{R^4}{4} - \frac{0^4}{4}\right] = \frac{\pi}{2}\rho L R^4$$

We are not done, because ρ is not one of our known variables. So, now we go back to the general equation for volumetric mass density and instead look at the density of the entire uniform solid cylinder:

$$\rho = \frac{m}{V} = \frac{m_{total}}{V_{total}} = \frac{M}{\pi R^2 L}$$

Now we have an equation for ρ which we can substitute back into our moment of inertia equation:

$$\Rightarrow I_z = \frac{\pi}{2}\left(\frac{M}{\pi R^2 L}\right) L R^4 \Rightarrow I_z = \frac{1}{2} M R^2$$

And notice the rotational inertia of a uniform solid cylinder about its long cylindrical axis does not depend on its density or length. The only two variables which affect its rotational inertia are mass and radius.

[1] http://www.flippingphysics.com/rotational-inertia-thin-hoop.html
[2] https://www.flippingphysics.com/thin-rod-rotational-inertia.html