Flipping Physics Lecture Notes:

Cross Product Torque (with a Cross Product Review)

http://www.flippingphysics.com/torque-cross-product.html

We have already learned an equation for the magnitude of torque[1]: $\tau = rF\sin\theta$

- F is the magnitude of the force causing the torque.
- r is the magnitude of the position vector from the axis of rotation to where the force acts on the object.
- θ is the angle between the direction of the force and the direction of r.
- And we can find the direction of the torque using the right-hand rule.[2]

A more general equation for torque uses the cross product: $\vec{\tau} = \vec{r} \times \vec{F}$

- **F** is the force causing the torque.
- **r** is the position vector from the axis of rotation to where the force is applied to the object.

Please recall that the cross product is **not** commutative. In other words: $\vec{\tau} = \vec{r} \times \vec{F} \neq \vec{F} \times \vec{r}$

However, the following is true: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Typically it has been a bit since y'all have used the cross product, so it is helpful to review how the cross product works.

Example #1: $\vec{A} = -\hat{i} + \hat{j} - 2\hat{k} \;\; \& \;\; \vec{B} = 2\hat{i} - 3\hat{j} + 4\hat{k} \Rightarrow \vec{A} \times \vec{B} = ?$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 2 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix}\hat{i} - \begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix}\hat{j} + \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix}\hat{k}$$

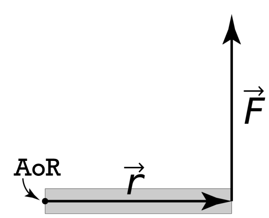
(I show the three 2x2 determinant matrices this one time to help review, however, I do not feel that it is necessary to show that step in the future.)

$$\Rightarrow \vec{A} \times \vec{B} = [(1)(4) - (-2)(-3)]\,\hat{i} - [(-1)(4) - (-2)(2)]\,\hat{j} + [(-1)(-3) - (1)(2)]\,\hat{k}$$

$$\Rightarrow \vec{A} \times \vec{B} = -2\hat{i} - 0\hat{j} + \hat{k} = -2\hat{i} + \hat{k}$$

Now let's return this back to torque and do some simple examples.

Example #2: $\vec{r} = \hat{i}\, m \;\; \& \;\; \vec{F} = \hat{j}\, N \Rightarrow \vec{\tau} = \vec{r} \times \vec{F} = ? = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$



$$\Rightarrow \vec{\tau} = [(0)(0) - (0)(1)]\,\hat{i} - [(1)(0) - (0)(0)]\,\hat{j} + [(1)(1) - (0)(0)]\,\hat{k} = \hat{k}\; N \cdot m$$

Recall that the cross product represents the area of the parallelogram created by the two vectors. Yes, cross product is a vector, therefore, the area has a direction. ☺

Notice that instead of using the cross product, we could use the magnitude torque equation:

$$\tau = rF\sin\theta = (1)(1)\sin(90^\circ) = 1\; N \cdot m \;\; \& \;\; \text{Right-Hand Rule} \Rightarrow \vec{\tau} = \hat{k}\; N \cdot m$$

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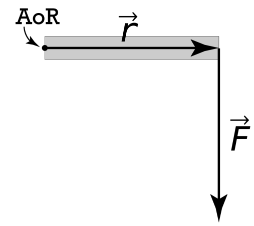
[1] https://www.flippingphysics.com/torque.html

[2] https://www.flippingphysics.com/right-hand-rule-torque.html

But we have to use the right-hand rule to determine the direction of the torque which can sometimes be inexact. When we use the cross product equation for torque, we get an exact direction for the torque.

Here is how we use the right-hand rule in this case. With your paper flat on your desk, starting with the fingers of your right hand at the axis of rotation, fingers point to the right in the direction of **r**, fingers curl 90° away from you in the direction of **F**, and your thumb points up or out of the page (or out of the board/out of the screen). Up is positive, therefore, out of the board or out of the screen is also positive. Up is the positive k direction.



Example #3:

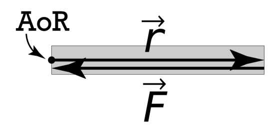
$$\vec{r} = \hat{i}\,m \;\&\; \vec{F} = -\hat{j}\,N \Rightarrow \vec{\tau} = \vec{r}\times\vec{F} = ? = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$\Rightarrow \vec{\tau} = [(0)(0)-(0)(-1)]\,\hat{i} - [(1)(0)-(0)(0)]\,\hat{j} + [(1)(-1)-(0)(0)]\,\hat{k} = -\hat{k}\,N\cdot m$$

Or we can again use the magnitude torque equation and the right-hand rule:

$$\tau = rF\sin\theta = (1)(1)\sin(90^\circ) = 1\,N\cdot m \;\&\; \text{Right-Hand Rule} \Rightarrow \vec{\tau} = -\hat{k}\,N\cdot m$$

With your paper flat on your desk, starting with the fingers of your right hand at the axis of rotation, fingers point to the right in the direction of **r**, fingers curl 90° toward you in the direction of **F**, and your thumb points down or into the page (or into the board/into the screen). Down is negative, therefore, into the board or into the screen is also negative. Down is the negative k direction.



Example #4:

$$\vec{r} = \hat{i}\,m \;\&\; \vec{F} = -\hat{i}\,N \Rightarrow \vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow \vec{\tau} = [(0)(0)-(0)(0)]\,\hat{i} - [(1)(0)-(0)(-1)]\,\hat{j} + [(1)(0)-(0)(-1)]\,\hat{k} = 0$$

Or we can use the equation for the magnitude of torque:

$$\tau = rF\sin\theta = rF\sin(180^\circ) = 0$$

And, bringing it back to the first example, only making it torque:

Example #1(b):

$$\vec{r} = \left[-\hat{i} + \hat{j} - 2\hat{k}\right] m \;\&\; \vec{F} = \left[2\hat{i} - 3\hat{j} + 4\hat{k}\right] N \Rightarrow \vec{\tau} = \vec{r}\times\vec{F} = ?$$

$$\vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 2 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix}\hat{i} - \begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix}\hat{j} + \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix}\hat{k}$$

$$\Rightarrow \vec{r}\times\vec{F} = [(1)(4)-(-2)(-3)]\,\hat{i} - [(-1)(4)-(-2)(2)]\,\hat{j} + [(-1)(-3)-(1)(2)]\,\hat{k}$$

$$\Rightarrow \vec{r}\times\vec{F} = -2\hat{i} - 0\hat{j} + \hat{k} = \left[-2\hat{i} + \hat{k}\right] N\cdot m$$

Notice this is a three dimensional problem where, if we were to use the magnitude torque equation instead of the cross product torque equation, we would need to use the Pythagorean theorem in three dimensions to determine the magnitudes of both vectors, and then we would have to determine the angle between those two vectors and then somehow use the right-hand rule to approximate the direction of the torque. All that sounds really complicated. However, because we used the cross product, the solution is rather straightforward, and we get an exact direction for the direction of the torque.