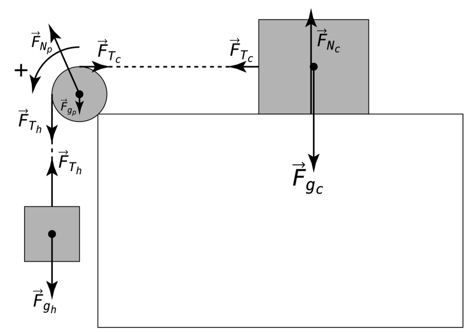
Flipping Physics Lecture Notes:

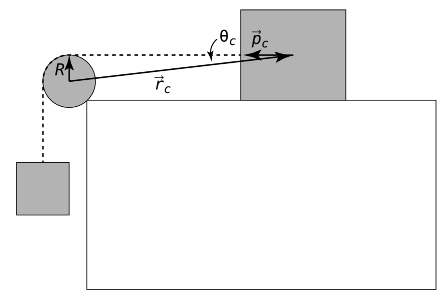
Angular Momentum and a Pulley Mass System
http://www.flippingphysics.com/angular-momentum-pulley-mass.html




Example: A cart of mass $m_c$ on a frictionless surface is attached via a string over a pulley to a hanging mass $m_h$. The frictionless pulley has radius R, mass $m_p$, and rotational inertia about its axis of $I_p$ equal to $xm_pR^2$. Determine the acceleration of the system.

knowns: $m_p, m_h, m_c, R_p = R, \mu_k = 0, I_p = xm_pR^2$ & $a = ?$

We start by identifying the forces in the free body diagrams of the pulley, the hanging mass, and the cart.




Let's determine the angular momentum of each object about the axis of rotation of the pulley.

$$L_p = I_p\omega_p = \left(xm_pR^2\right)\omega_p = xm_pR^2\left(\frac{v_t}{R}\right) = xm_pRv_t \;\&\; v_t = r\omega \Rightarrow \omega_p = \frac{v_t}{R}$$

$$L_c = r_c \times p_c = r_c m_c v_c \sin\theta_c = Rm_cv_c$$

$$\text{where } \sin\theta_c = \frac{O}{H} = \frac{R}{r_c} \Rightarrow R = r_c\sin\theta_c \text{ and } v_t = v_h = v_c = v$$

$$L_h = r_h \times p_h = r_h m_h v_h \sin\theta_h = Rm_hv_h$$

$$L_{\text{total}} = L_p + L_h + L_c = xm_pRv + Rm_hv + Rm_cv = Rv\left(xm_p + m_h + m_c\right)$$

$$\sum_{\substack{\text{AOR Pulley}\\ \text{whole system}}} \tau = +\tau_{F_{g_h}} - \tau_{F_{T_h}} + \tau_{F_{T_h}} + \tau_{F_{gp}} + \tau_{F_{N_p}} - \tau_{F_{T_c}} + \tau_{F_{T_c}} - \tau_{F_{gc}} + \tau_{F_{N_c}}$$

Positive direction is defined as shown: Down is positive at hanging mass. Left is positive at cart.

Using the right-hand rule to determine positive and negative directions. Torque caused by the force of …:

- gravity on the hanging mass is positive.
- tension on the hanging mass is negative.
- tension from the hanging mass acting on the pulley is positive.
- gravity acting on the pulley is zero because the force acts at the axis of rotation and r is therefore zero.
  - r in the equation $\tau = rF\sin\theta$
  - r is the distance from the axis of rotation to where the force acts on the object.
- normal acting on the pulley is zero because the force acts at the axis of rotation and r is therefore zero.
- tension from the cart acting on the pulley is negative.
- tension on the cart is positive.
- gravity on the cart is negative.
- normal on the cart is positive.

Torques we can cancel out:

- The two torques caused by the forces of tension from the hanging mass are equal and opposite and cancel one another out.
- The two torques caused by the forces of tension from the cart are equal and opposite and cancel one another out.
- Acceleration of the cart in the y-direction is zero, therefore, the net force in the y-direction on the cart is zero, and the force of gravity acting on the cart and the force normal acting on the cart are of equal magnitude. And because both torques have the same magnitude lever arm, the minimum distance from the axis of rotation to the line of action of the force, those two torques are also equal and opposite, and cancel one another out.

The only torque which remains is the torque caused by the force of gravity acting on the hanging mass:

$$\Rightarrow \sum \tau = +\tau_{F_{g_h}} = \frac{dL_{total}}{dt} \Rightarrow r_h F_{g_h} \sin\theta_h = \frac{d}{dt}\left(Rv\left(xm_p + m_h + m_c\right)\right)$$

$$\Rightarrow Rm_h g = R\left(xm_p + m_h + m_c\right)\left(\frac{d}{dt}(v)\right) \Rightarrow m_h g = \left(xm_p + m_h + m_c\right) a \Rightarrow \boxed{a = \frac{m_h g}{xm_p + m_h + m_c}}$$

It is important to realize that, for the pulley, this is the linear acceleration of the rim of the pulley, at any radius smaller than R, the linear acceleration of that part of the pulley is smaller than this value.

Alternatively, we could have solved this by summing the torques on just the pulley, and summing the forces on each mass separately:

$$\sum_{\substack{\text{AOR Pulley} \\ \text{pulley only}}} \tau = \tau_{F_{T_h}} + \tau_{F_{gp}} + \tau_{F_{Np}} - \tau_{F_{T_c}} = \frac{dL_p}{dt} \Rightarrow \tau_{F_{T_h}} - \tau_{F_{T_c}} = \frac{d}{dt}\left(xm_p R v\right)$$

$$\Rightarrow r_h F_{T_h} \sin\theta_h - r_c F_{T_c} \sin\theta_c = \left(xm_p R\right)\frac{d}{dt}(v)$$

$$\Rightarrow RF_{T_h} \sin 90^\circ - RF_{T_c} \sin 90^\circ = xm_p Ra \Rightarrow F_{T_h} - F_{T_c} = xm_p a$$

$$\sum_{\substack{\text{hanging mass} \\ \text{+direction}}} F = F_{g_h} - F_{T_h} = m_h a \Rightarrow m_h g - F_{T_h} = m_h a \Rightarrow F_{T_h} = m_h g - m_h a$$

$$\sum_{\substack{\text{cart} \\ \text{+direction}}} F = F_{T_c} = m_c a$$

$$F_{T_h} - F_{T_c} = xm_p a \Rightarrow (m_h g - m_h a) - m_c a = xm_p a$$

$$\Rightarrow m_h g = xm_p a + m_h a + m_c a = \left(xm_p + m_h + m_c\right) a \Rightarrow \boxed{a = \frac{m_h g}{xm_p + m_h + m_c}}$$

Or we could have used conservation of mechanical energy instead:
Initial point where the everything starts. Final point after cart and hanging mass have moved a distance y. Horizontal zero line at the final height of the hanging mass. Cart and Pulley have no change in mechanical energy.

$$ME_i = ME_f \Rightarrow m_h g h_{h_i} = \frac{1}{2} I_p \omega_f^2 + \frac{1}{2} m_h v_f^2 + \frac{1}{2} m_c v_f^2$$

$$\Rightarrow m_h g y = \frac{1}{2}\left(xm_p R^2\right)\left(\frac{v_f}{R}\right)^2 + \frac{1}{2} m_h v_f^2 + \frac{1}{2} m_c v_f^2$$

$$\Rightarrow 2m_h g y = xm_p v_f^2 + m_h v_f^2 + m_c v_f^2 = \left(xm_p + m_h + m_c\right) v_f^2$$

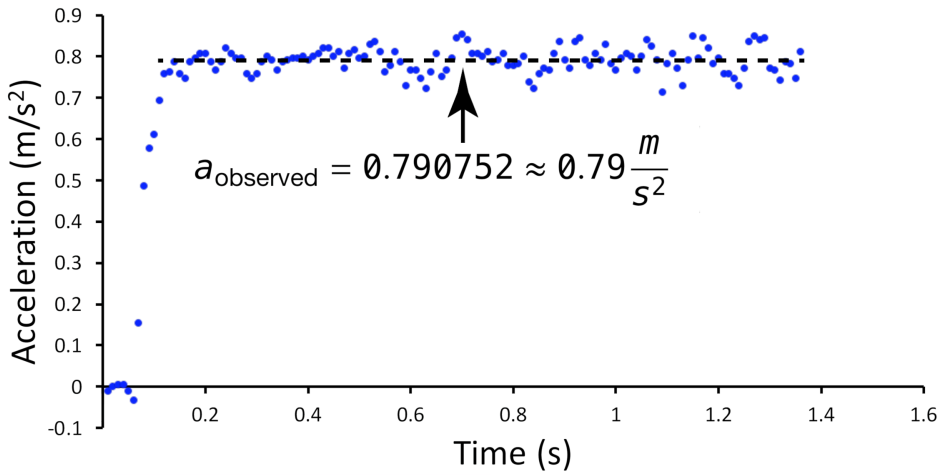
$$\Rightarrow v_f^2 = \frac{2m_h g y}{xm_p + m_h + m_c} \quad \& \quad v_f^2 = v_i^2 + 2a\Delta x \Rightarrow \frac{2m_h g y}{xm_p + m_h + m_c} = 0 + 2ay$$

$$\Rightarrow \boxed{a = \frac{m_h g}{xm_p + m_h + m_c}}$$

And, of course, we can test this by doing the experiment:[1]

knowns: $m_p = 3.62\,g,\ m_h = 25.0\,g,\ m_c = 271\,g,\ x = 0.91$

$$a = \frac{(25)(9.81)}{(0.91)(3.62) + 25 + 271} = 0.819353 \approx 0.82\frac{m}{s^2}$$




$$a_{\text{observed}} = 0.790752 \approx 0.79\ \frac{m}{s^2}$$

$$E_r = \frac{O - A}{A} \times 100 = \frac{0.790752 - 0.819353}{0.819353} \times 100 = -3.49950 \approx -3.5\%$$

The Physics Works!!!

(The most likely culprit for the error is friction between the cart and the track.)

[1] Yes, I sacrificed a Pasco Smart Pulley to measure the mass of the pulley wheel. (I manually separated the pulley wheel from the rest of the smart pulley using a saw so I could measure the mass of the pulley. ☺) Also, The rotational inertia and radius of the smart pulley are given here: ftp://ftp.pasco.com/support/documents/english/me/ME-9341/012-03051F.pdf That, combined with the measured mass of the smart pulley shown on page 3 of these lecture notes, provided the information to be able to determine that x = 0.91.