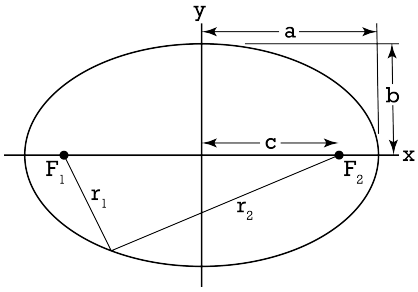
Flipping Physics Lecture Notes:

Kepler's Third Law Derivation
http://www.flippingphysics.com/kepler-third-derivation.html

Kepler's Third Law of Planetary Motion:
"The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit; in other words, $T^2 \propto a^3$."

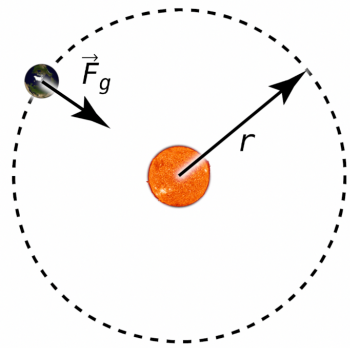


To prove Kepler's Third Law we start by assuming the orbit of the planet is circular.[1] The only force acting on the planet is the force of gravity between the Sun and the planet. Therefore:

$$\sum \vec{F}_{\text{in}} = \vec{F}_g = m\vec{a}_c \Rightarrow \frac{Gm_s m_p}{r^2} = m_p r\omega^2 \Rightarrow Gm_s = r^3\omega^2$$

$$\& \quad \omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} \Rightarrow Gm_s = r^3\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 r^3}{T^2}$$

$$\Rightarrow T^2 = \left(\frac{4\pi^2}{Gm_s}\right) r^3$$



It is important to realize we have proved Kepler's Third Law for circular orbits; however, we have not explicitly proved it for elliptical orbits.

[1] We have shown previously that the orbits of all the planets are nearly circular. See "Kepler's First Law of Planetary Motion": http://www.flippingphysics.com/kepler-first.html