Flipping Physics Lecture Notes:

Universal Gravitational Potential Energy Derivation
http://www.flippingphysics.com/universal-gravitational-potential-energy-derivation.html

We have already learned about Universal Gravitational Potential Energy[1]; today we derive the equation. To do that, we need to remember that the force of gravity is a conservative force.[2] In other words, the work done by the force of gravity on an object is independent of the path taken by the object.

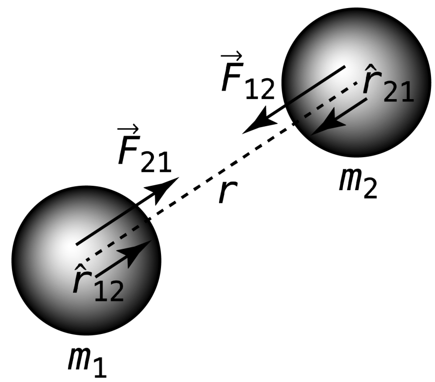
$$U_g = -\frac{Gm_1m_2}{r}$$

An equation which is always true for a conservative forces is: $F_{\text{conservative}} = -\frac{\mathrm{d}U}{\mathrm{d}x} \Rightarrow F_g = -\frac{\mathrm{d}U_g}{\mathrm{d}r}$

Which we can rearrange: $F_g = -\frac{\mathrm{d}U_g}{\mathrm{d}r} \Rightarrow \mathrm{d}U_g = -F_g\mathrm{d}r \Rightarrow \int_{U_i}^{U_f} \mathrm{d}U_g = -\int_{r_i}^{r_f} F_g\mathrm{d}r$

$$\Rightarrow \left[U_g\right]_{U_i}^{U_f} = U_{g_f} - U_{g_i} = \Delta U_g = -\int_{r_i}^{r_f} F_g\mathrm{d}r = -W_{F_g} \Rightarrow W_{\text{conservative force}} = -\Delta U$$

We have just shown (again) that the work done by a conservative force equals the negative of the change in potential energy associated with the force. And now returning from our brief detour…



Because the force of gravity is an attractive force, the vector universal gravitational force equation is: $\vec{F}_{12} = -\frac{Gm_1m_2}{r^2}\hat{r}_{12}$

Which we can substitute back into our previous equation and do some math:

$$\Rightarrow \Delta U_g = -\int_{r_i}^{r_f} \left(-\frac{Gm_1m_2}{r^2}\right)\mathrm{d}r = Gm_1m_2\int_{r_i}^{r_f} \frac{1}{r^2}\mathrm{d}r$$

Another way to understand why the negative is in front of the force of gravity under the integral above is that work equals the integral of the dot product of force with respect to position. Those two vectors are opposite in direction (**F**g and d**r**) and that gives an angle of 180°, and the cosine of 180° is negative 1:

And now returning back to the original solution…

$$\Rightarrow \Delta U_g = Gm_1m_2\int_{r_i}^{r_f} r^{-2}\mathrm{d}r = Gm_1m_2\left[\frac{r^{-1}}{-1}\right]_{r_i}^{r_f} = -Gm_1m_2\left[\frac{1}{r}\right]_{r_i}^{r_f}$$

$$\Rightarrow \Delta U_g = -Gm_1m_2\left[\frac{1}{r_f} - \frac{1}{r_i}\right] \text{ and let } U_{g_i} = 0 \text{ where } F_g = 0 \rightarrow r_i = \infty$$

$$\Rightarrow U_{g_f} - U_{g_i} = -Gm_1m_2\left[\frac{1}{r_f} - \frac{1}{\infty}\right] \Rightarrow U_{g_f} - 0 = -Gm_1m_2\left[\frac{1}{r_f} - 0\right] \Rightarrow U_g = -\frac{Gm_1m_2}{r}$$

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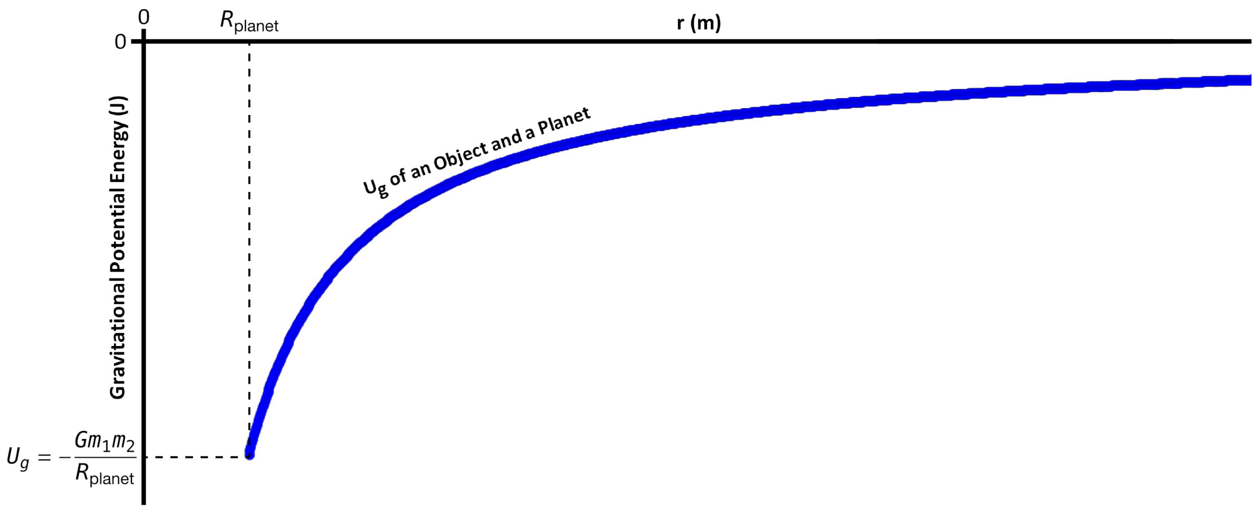
[1] Universal Gravitational Potential Energy: http://www.flippingphysics.com/universal-gravitational-potential-energy.html
[2] Conservative Force and Potential Energy: http://www.flippingphysics.com/conservative-force-energy.html

That means the gravitational potential energy which exists between an object and a planet is:

$U_g = -\frac{Gm_o m_p}{r}$ on planet surface $U_g = -\frac{Gm_o m_p}{R_p}$



If you want a detailed look at the force of gravity and gravitational potential energy which exist between an object and a planet all the way from the center of the planet to infinitely far away, please see my lesson "Force of Gravity and Gravitational Potential Energy Functions from Zero to Infinity (but not beyond)".[3]

[3] http://www.flippingphysics.com/gravity-zero-infinity.html