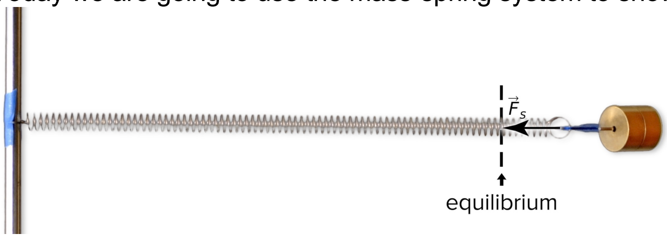
Flipping Physics Lecture Notes:
Simple Harmonic Motion Derivations using Calculus (Mass-Spring System)
http://www.flippingphysics.com/SHM-derivation-mass-spring.html

As we have discussed before[1], a mass-spring spring system is an example of simple harmonic motion. Today we are going to use the mass-spring system to show a calculus approach to simple harmonic motion.



An object is in simple harmonic motion if the acceleration of the object is proportional to its displacement from equilibrium and directed opposite the object's displacement from equilibrium. The force from a spring[2] can be the restoring force which causes this acceleration. $F_{\text{spring}} = -kx$ When the mass is displaced to the right from equilibrium, the force of the spring is to the left and:

$$\sum F_x = F_{\text{spring}} = ma_x \Rightarrow -kx = ma_x \Rightarrow a_x = -\left(\frac{k}{m}\right)x \Rightarrow a_{\max} = \left(\frac{k}{m}\right)A$$

A is amplitude and is the maximum distance from equilibrium.

We can use the derivative definitions of acceleration and velocity to get a relationship between acceleration and position:

$$a = \frac{dv}{dt} \ \& \ v = \frac{dx}{dt} \Rightarrow a = \frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

The following is a mathematical definition of simple harmonic motion. If the motion of an object satisfies this equation, then the motion of the object is simple harmonic motion:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

The mass-spring system satisfies this equation with $\omega^2 = \dfrac{k}{m}$ where ω is "angular frequency".

A position equation which satisfies the mathematical definition of simple harmonic motion is:

$$x(t) = A\cos(\omega t + \phi)$$

Here we show that the above equation satisfies the mathematical definition of simple harmonic motion:

$$v(t) = \frac{dx}{dt} = \frac{d}{dt}\left[A\cos(\omega t + \phi)\right] = A\frac{d}{dt}\left[\cos(\omega t + \phi)\right] = A\left[-\sin(\omega t + \phi)\right]\left[\frac{d}{dt}(\omega t + \phi)\right]$$

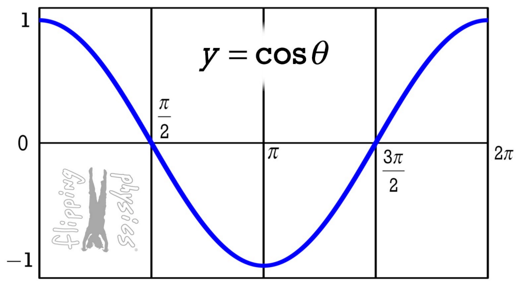
$$\Rightarrow v(t) = -A\left[\sin(\omega t + \phi)\right](\omega) = -A\omega\sin(\omega t + \phi) \Rightarrow v_{\max} = A\omega$$

$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\left[-A\omega\sin(\omega t + \phi)\right] = -A\omega\frac{d}{dt}\left[\sin(\omega t + \phi)\right] = -A\omega\left[\cos(\omega t + \phi)\right]\left[\frac{d}{dt}(\omega t + \phi)\right]$$

$$\Rightarrow a(t) = -A\omega\left[\cos(\omega t + \phi)\right](\omega) \Rightarrow a(t) = -A\omega^2\cos(\omega t + \phi) \Rightarrow a_{\max} = A\omega^2$$

$$\Rightarrow a(t) = -\omega^2\left[A\cos(\omega t + \phi)\right] \Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x$$

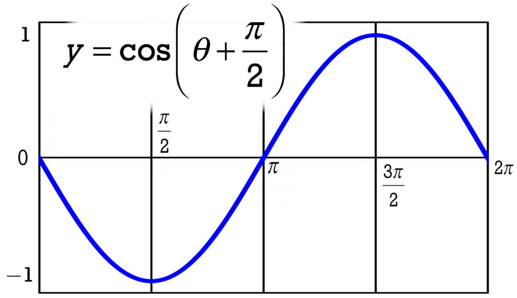
[1] "Simple Harmonic Motion Position Equation Derivation" https://www.flippingphysics.com/shm-position.html
[2] The negative in Hooke's Law (spring force equation) indicates the direction of the spring force is opposite the displacement from equilibrium.



$\phi = $ **Phase Constant**

In this example: $\phi = \frac{\pi}{2}\, radians = 90^\circ$

The phase constant phi, ϕ:
- or phase shift, phase angle, phase delay
- Shifts wave location along the horizontal axis
- Does not change the shape of the wave



Please remember to use radians for ϕ, the phase constant, and radians per second for angular frequency! Speaking of angular frequency…

Period, T, is the time interval it takes for a particle to go through one full cycle. In other words, the values of x should be equal at times t and t + T. Also, after one full cycle the phase has increased by 2π radians. In other words:

$$x(t) = x(t+T) \Rightarrow A\cos(\omega t + 2\pi) = A\cos[\omega(t+T) + 0]$$

$$\omega t + 2\pi = \omega t + \omega T \Rightarrow 2\pi = \omega T \Rightarrow T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T}$$

$$f = \frac{1}{T} \Rightarrow \omega = \frac{2\pi}{T} = 2\pi f$$

Notice angular frequency ω and frequency f are not the same. The units on angular frequency are radians per second and the units on frequency are one over seconds or cycles per second.

Previously we solved for angular frequency this way:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T}$$

This provides the same result; however, it does not involve using the equation for angular velocity.

Going back to the mass-spring system specifically, we can solve for the period of a mass-spring system:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

Notice all these equations are general equations for Simple Harmonic Motion. You just need to derive the angular frequency for each new physical arrangement.

Condition for SHM is $\frac{d^2x}{dt^2} = -\omega^2 x$ & ω is angular frequency & $T = \frac{2\pi}{\omega}$

$$x(t) = A\cos(\omega t + \phi) \Rightarrow x_{max} = A$$

$$v(t) = -A\omega\sin(\omega t + \phi) \Rightarrow v_{max} = A\omega$$

$$a(t) = -A\omega^2\cos(\omega t + \phi) \Rightarrow a_{max} = A\omega^2$$