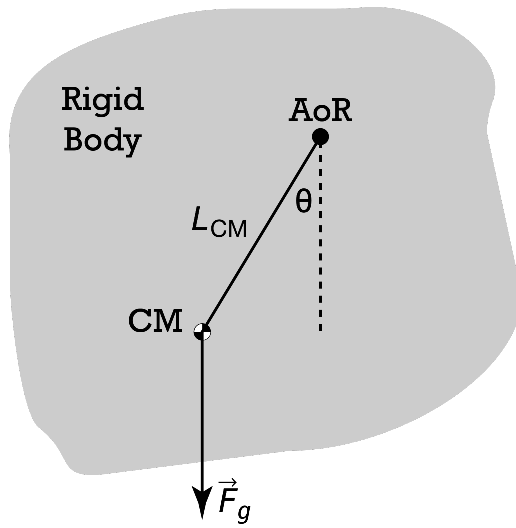
Flipping Physics Lecture Notes:

Physical Pendulum - Period Derivation and Demonstration using Calculus
http://www.flippingphysics.com/physical-pendulum.html

As we have discussed before[1], the mathematical condition for simple harmonic motion is: $\frac{d^2x}{dt^2} = -\omega^2 x$



And we have derived the period of a simple pendulum.[2]

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Today we are going to derive the period of a physical pendulum. Recall that a simple pendulum is defined as an ideal pendulum where all the mass of the pendulum is concentrated in the pendulum bob. In other words, the pendulum bob is considered to be a point mass, and the string is massless and inextensible[3], and there is, of course, no friction.

A physical pendulum is a rigid body suspended by a single point which oscillates without friction. Therefore, there is a force of gravity acting on the center of mass of the rigid body which causes a torque around the axis of rotation. The net torque around the axis of rotation is:

$$\sum \tau = \tau_{F_g} = I\alpha \Rightarrow -rF_g \sin\theta = -L_{CM} mg \sin\theta = I\frac{d^2\theta}{dt^2} \Rightarrow \frac{d^2\theta}{dt^2} = -\frac{mgL_{CM}}{I}\sin\theta$$

And, according to the small angle approximation we now have: $\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{mgL_{CM}}{I}\theta$

$\sin\theta \approx \theta$ (in radians)

Going back to the definition of simple harmonic motion, we can solve for the angular frequency of a physical pendulum:

$$\frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \frac{d^2\theta}{dt^2} = -\omega^2\theta \Rightarrow \omega^2 = \frac{mgL_{CM}}{I} \Rightarrow \omega = \sqrt{\frac{mgL_{CM}}{I}}$$

Which we can use to solve for the period of a physical pendulum:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{mgL_{CM}}{I}}} \Rightarrow T = 2\pi\sqrt{\frac{I}{mgL_{CM}}}$$

Notice what happens when we assume the physical pendulum is actually a simple pendulum:

$$I_{\text{point mass}} = mr^2 \Rightarrow T = 2\pi\sqrt{\frac{mr^2}{mgL_{CM}}} = 2\pi\sqrt{\frac{L^2}{gL}} = 2\pi\sqrt{\frac{L}{g}}$$

We get the equation for the period of a simple pendulum. ☺

[1] "Simple Harmonic Motion Derivations using Calculus (Mass-Spring System)" http://www.flippingphysics.com/SHM-derivation-mass-spring.html
[2] "Simple Pendulum - Simple Harmonic Motion Derivation using Calculus" http://www.flippingphysics.com/SHM-derivation-pendulum.html
[3] "inextensible" simply means the string has a constant length.

All the motion equations we solved for last time to describe the simple harmonic motion of a simple pendulum are valid for a physical pendulum, it is just that the angular frequency has changed:

$$\theta(t) = \theta_{max}\cos(\omega t + \phi) \Rightarrow \theta_{max} = \theta_{max}$$

$$\omega(t) = \frac{d\theta}{dt} = -\theta_{max}\omega\sin(\omega t + \phi) \Rightarrow \omega_{max} = \theta_{max}\omega$$

$$\alpha(t) = \frac{d\omega}{dt} = -\theta_{max}\omega^2\cos(\omega t + \phi) \Rightarrow \alpha_{max} = \theta_{max}\omega^2$$

And we can, of course, make sure the physics works! Let's use a long, thin, uniform rod. We already derived the rotational inertia of a long, thin, uniform rod:

$$I_{end} = \frac{1}{3}ML^2 \;\;\&\;\; T = 2\pi\sqrt{\frac{I}{mgL_{CM}}} \Rightarrow \frac{T^2}{4\pi^2} = \frac{I}{mgL_{CM}} \Rightarrow I = \frac{T^2 mgL_{CM}}{4\pi^2}$$

knowns: $M = 0.667\ kg \;\&\; L = 0.909\ m$

$$I_{end} = \frac{1}{3}ML^2 = \frac{1}{3}(0.667)(0.909)^2 = 0.18371 \approx 0.184\ kg \cdot m^2$$

$$T = \frac{15.63\text{sec}}{10\text{cycles}} = 1.563\frac{\text{sec}}{\text{cycle}}$$

$$I = \frac{T^2 mgL_{CM}}{4\pi^2} = \frac{(1.563)^2(0.667)(9.81)\left(\frac{0.909}{2}\right)}{4\pi^2} = 0.18403 \approx 0.184\ kg \cdot m^2$$

$$E_r = \frac{O - A}{A}\times 100 = \frac{(0.18403) - (0.18371)}{0.18371}\times 100 = 0.17360 \approx 0.174\%$$

The Physics Works!!!