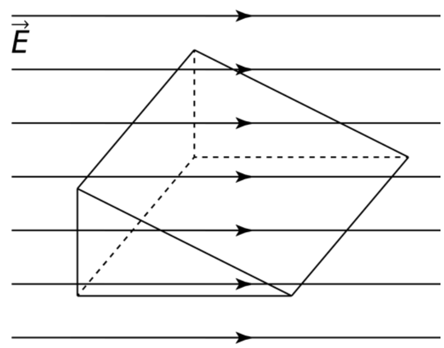
Flipping Physics Lecture Notes:
Electric Flux and Gauss' Law
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-electric-flux-gauss-law.html

Flux is defined as any effect that appears to pass or travel through a surface or substance, however, realize that effect does not need to move. Hence, "appears to".

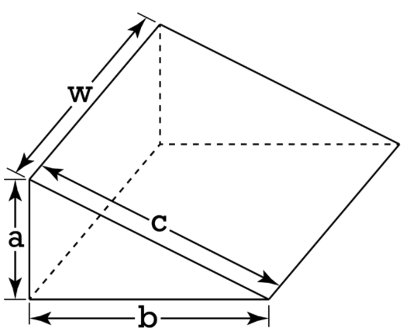
Electric flux is the measure of the amount of electric field which passes through a defined area. The equation for electric flux of a uniform electric field is:

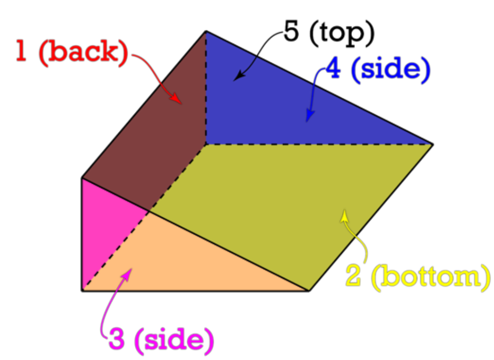
- $\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta$
- Φ is the uppercase, Greek letter phi
- E is the uniform electric field *(use magnitude)*
- A is the area of the surface through which the uniform electric field is passing *(use magnitude)*
- θ is the angle between the directions of E and A
  - Notice this is the same form as the equation for work. This means you use the magnitudes of E and A, and cosθ determines if the electric flux is positive or negative
  - $W = \vec{F} \cdot \Delta\vec{r} = F\Delta r\cos\theta$
- Electric flux is a scalar
- The units for electric flux are $\frac{N \cdot m^2}{C}$



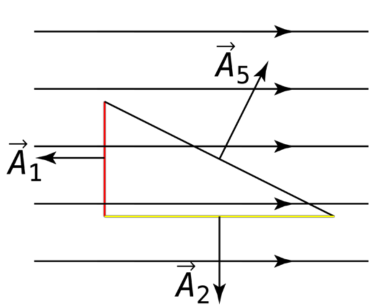
Usually, electric flux is through some sort of closed surface. So, let's do an example and determine the net electric flux of a uniform, horizontal electric field through a right triangular box.

Let's define and label the dimensions and sides of the triangular box as:





And now we can determine the electric flux through each side:



Electric flux for Area 1 (back): θ_1 is 180° because Area 1 is to the left or out of the rectangular box and the electric field is to the right.

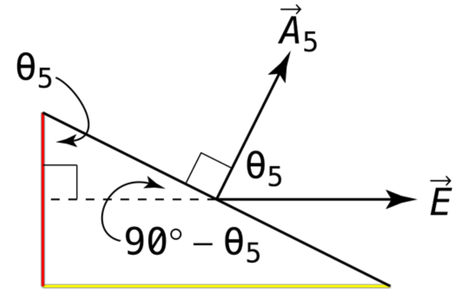
$$\Phi_1 = EA_1\cos\theta_1 = E(aw)\cos(180°) = -Eaw$$

Electric flux for Area 2 (bottom): θ_2 is 90° because Area 2 is down or out of the rectangular box and the electric field is to the right.

$$\Phi_2 = EA_2\cos\theta_2 = E(bw)\cos(90°) = 0$$

Electric flux for Areas 3 and 4 (sides): θ_3 and θ_4 are both 90° because Area 3 is out of the page and Area 4 is into the page (and the electric field is to the right).

$$\Phi_3 = EA_3 cos\theta_3 = E\left(\frac{1}{2}ba\right)\cos(90^\circ) = 0 = \Phi_4$$



Electric flux for Area 5 (top): To understand why $\cos\theta_5 = a/c$, we need to draw another diagram.

$$\cos\theta_5 = \frac{A}{H} = \frac{a}{c}$$

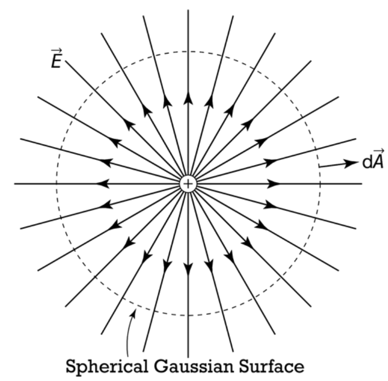
$$\Phi_5 = EA_5 cos\theta_5 = E(cw)\left(\frac{a}{c}\right) = Eaw$$

And the total electric flux through the entire triangular box is:

$$\Phi_{total} = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5 = -Eaw + 0 + 0 + 0 + Eaw = 0$$

Notice that:

- When an electric field is going into a closed surface, the electric flux is negative.
- When an electric field is coming out of a closed surface, the electric flux is positive.



Let's now do another example. Let's determine the electric flux passing through a sphere which is concentric to and surrounds a positive point charge.

Notice we cannot use $\Phi_E = \vec{E}\cdot\vec{A} = EAcos\theta$ because the electric field is not uniform. We need to use the integral equation for electric flux:

$$\Phi_E = \vec{E}\cdot\vec{A} \Rightarrow d\Phi_E = \vec{E}\cdot d\vec{A} \Rightarrow \int d\Phi_E = \int \vec{E}\cdot d\vec{A} \Rightarrow \Phi_E = \int \vec{E}\cdot d\vec{A}$$

$$\Phi_E = \int \vec{E}\cdot d\vec{A} = \int E dA\cos\theta = \int E dA\cos(0) = \int E dA = E\int dA = EA$$

$$\vec{F}_{21} = k\frac{(q_1)(q_2)}{r^2}\hat{r}_{21} \;\&\; \vec{E} = \frac{\vec{F}_e}{q} \Rightarrow E_{+\text{ point charge}} = \frac{\frac{kqQ}{r^2}}{q} = \frac{kQ}{r^2} \;\&\; A_{sphere} = 4\pi r^2$$

$$\Rightarrow \Phi_E = \left(\frac{kQ}{r^2}\right)\left(4\pi r^2\right) = 4\pi kQ = 4\pi\left(\frac{1}{4\pi\varepsilon_0}\right)Q = \frac{Q}{\varepsilon_0}$$

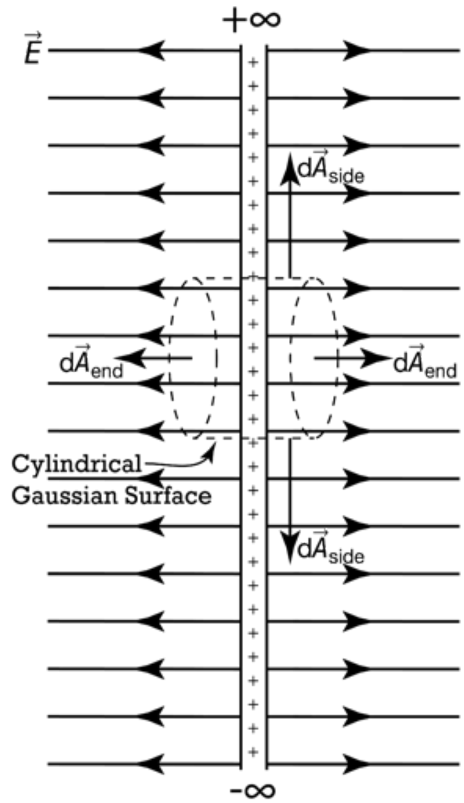
In other words, the electric flux through a closed Gaussian surface is:

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$$

This is Gauss' law!

Gauss' law relates electric flux through a Gaussian surface to the charge enclosed by the Gaussian surface:

- A Gaussian surface is a three-dimensional closed surface
- While the Gaussian surfaces we usually work with are imaginary, the Gaussian surface could actually be a real, physical surface
- Typically, we choose the shapes of our Gaussian surfaces such that the electric field generated by the enclosed charge is either perpendicular or parallel to the various sides of the Gaussian surface. This greatly simplifies the surface integral because all the angles are multiples of 90° and the cosine of those angles have a value of -1, 0, or 1.
- As long as the amount of charge enclosed in a Gaussian surface is constant, the total electric flux through the Gaussian surface does not depend on the size of the Gaussian surface.
- Gauss' law is the first of Maxwell's equations which are a collection of equations which fully describe electromagnetism.



Notice then that, if the net charge inside a closed Gaussian surface is zero, then the net electric flux through the Gaussian surface is zero.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0} = \frac{0}{\varepsilon_0} = 0$$

This is why the net electric flux through the closed rectangular box in our first example was zero.

Let's do another example using Gauss' law. Determine the electric field which surrounds an infinitely large, thin plane of positive charges with uniform surface charge density, σ:

First off, we know the electric field will be directed normal to and away from the infinite plane of positive charges. This is because the plane is infinitely large; therefore, every component of the electric field, dE, which is parallel to the plane of charges and is caused by infinitesimally small, charged pieces of the plane, dq, will cancel out leaving only electric field components of dE which are perpendicular to the plane and directed away from the plane.

We pick a Gaussian surface such that it is a cylinder with ends parallel to the plane of charges and a side parallel to the electric field and use Gauss' law. The two ends of the Gaussian cylinder are equidistant from the charged plane.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0}$$

$$\Phi_E = \int_{side} \vec{E} \cdot d\vec{A}_{side} + \int_{\substack{left\\end}} \vec{E} \cdot d\vec{A}_{end} + \int_{\substack{right\\end}} \vec{E} \cdot d\vec{A}_{end} = \frac{q_{in}}{\varepsilon_0}$$

$$\Rightarrow \Phi_E = \int_{side} E dA \cos\theta_{side} + \int_{\substack{left\\end}} E dA \cos\theta_{end} + \int_{\substack{right\\end}} E dA \cos\theta_{end} = \frac{q_{in}}{\varepsilon_0}$$

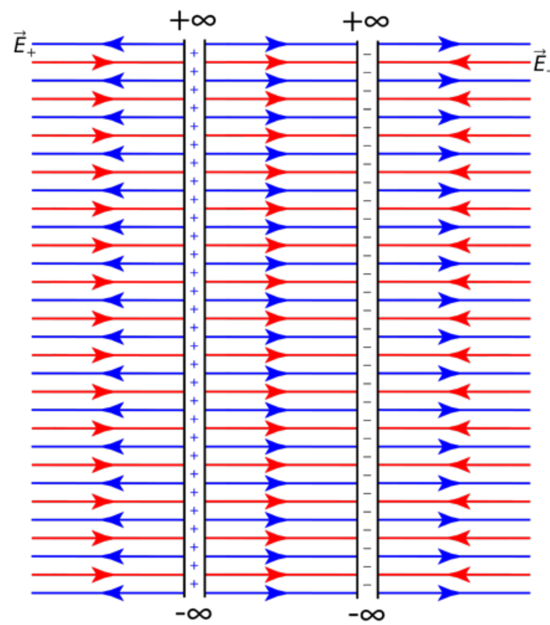
$$\Rightarrow \Phi_E = \int_{side} E dA \cos(90°) + \int_{\substack{left\\end}} E dA \cos(0°) + \int_{\substack{right\\end}} E dA \cos(0°) = \frac{q_{in}}{\varepsilon_0}$$

$$\Rightarrow \Phi_E = E \int_{\substack{\text{left}\\ \text{end}}} dA + E \int_{\substack{\text{right}\\ \text{end}}} dA = E\left(2A_{end}\right) = \frac{q_{in}}{\varepsilon_0}$$

$$\& \ \sigma = \frac{Q}{A} = \frac{q_{in}}{A_{end}} \Rightarrow q_{in} = \sigma A_{end} \Rightarrow E\left(2A_{end}\right) = \frac{\sigma A_{end}}{\varepsilon_0} \Rightarrow E = \frac{\sigma}{2\varepsilon_0}$$

Notice this electric field is uniform and is independent of the distance from the infinite plane of charges.

And notice what happens if we have two infinite parallel planes of charges, one with positive charge and one with negative charge:



The electric field outside the planes of charges cancels out to give zero electric field outside the planes of charges:

$$E_{outside} = 0$$

And between the two planes of charges, the electric fields add together:

$$E_{between} = 2E_{one\ plate} = 2\left(\frac{\sigma}{2\varepsilon_0}\right) = \frac{\sigma}{\varepsilon_0}$$

And we have begun our journey towards determining the capacitance of a parallel plate capacitor…

Notice that a positively charged particle moving through a constant electric field will experience an electrostatic force in the direction of the electric field. This force will be constant and equal to qE. In other words, the motion of a charged particle through a constant electric field will have similar characteristics to a mass moving through the constant gravitational field near the surface of a planet. This is very similar to projectile motion.

A few loose ends:

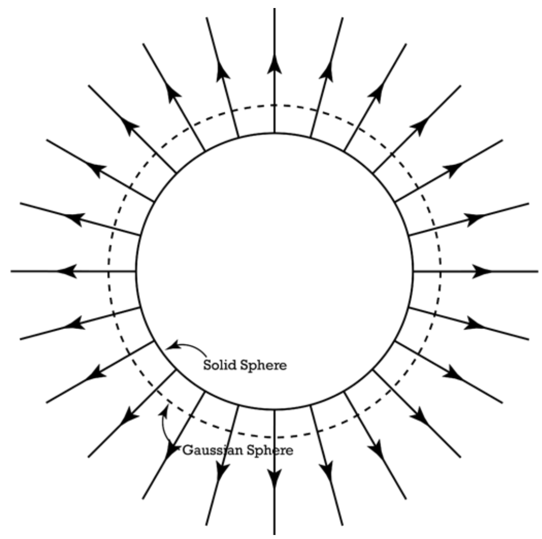
- According to the AP Physics C: Electricity and Magnetism Guidelines, you are responsible for a quantitative approach of Gauss' law to solve for electric fields only for charge distributions which are spherically, cylindrically, or planarly symmetric.
    a. In other words, to keep the math from getting overly complicated, you are responsible for solving equations with Gauss' law only in situations which are highly symmetrical.
    b. Some examples are:
        i. Inside and outside of a solid sphere made of conducting or insulating material. The Gaussian surface is a sphere.
        ii. Inside and outside of a thin spherical shell. Again, the Gaussian surface is a sphere.
        iii. An infinitely long, thin line of charges. The Gaussian surface is a cylinder that is colinear with the line of charges.
        iv. An infinitely large, thin plane of charges. We just did this. The Gaussian surface is a cylinder whose ends are parallel to the plane of charges.
        v. 2 infinitely large, thin parallel planes of charges. We just did this. Do a single plane of charges first.

- Do not forget the three kinds of charge densities which you are responsible for being able to use with Gauss' law depending on whether the charge is one, two, or three dimensional.
  a. linear charge density, $\lambda = \frac{Q}{L}$ in $\frac{C}{m}$
  b. surface charge density, $\sigma = \frac{Q}{A}$ in $\frac{C}{m^2}$
  c. volumetric charge density, $\rho = \frac{Q}{V}$ in $\frac{C}{m^3}$

Lastly, realize Gauss' law uses electric flux which is a measure of the number of imaginary electric field lines which pass through an imaginary Gaussian surface and those imaginary field lines are caused by highly symmetric groups of stationary point charges which are imperceptible to the naked eye. Yeah, it takes a bit of imagination to be able to visualize all of this. Which is why you need to practice!

A loose end:

- Outside the surface of a uniformly charged sphere, the electric field is the same as if the charged sphere were a point particle.
  - Example: Solid, uniformly charged sphere with charge Q and radius, a.
  - Create a Gaussian surface which is a concentric sphere with radius r > a.



$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_0}$$

$$\Rightarrow \Phi_E = \int_{sphere} E \cos\theta \, dA = \int_{sphere} E \cos(0°) \, dA = \frac{q_{enclosed}}{\varepsilon_0}$$

$$\Rightarrow E \int_{sphere} dA = EA_{sphere} = E\left(4\pi r^2\right) = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow E = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q}{r^2} \Rightarrow E = \frac{kQ}{r^2}$$

  - This is true of a conductor or an insulator, however, the electric field inside a conductor will be zero, and inside an insulator the electric field depends on the radius and charge distribution, and can be derived in a similar manner.