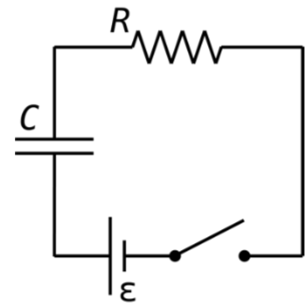
Flipping Physics Lecture Notes:
Charge and Current vs. Time in an RC Circuit
http://www.flippingphysics.com/rc-circuit-current-charge.html

We already determine the limits of charge and current when charging a capacitor in an RC circuit.



Now, let's figure out what happens between $t_i = 0$ and $t_f \approx \infty$, however, before we do, I want to point out that AP Physics C: Electricity and Magnetism students are responsible for knowing how to derive these equations. So, yes, you do need to understand these derivations and be able to do them on your own. And here we go … starting with our Kirchhoff's Loop Rule equation:

$$0 = \varepsilon - \frac{q}{C} - iR \Rightarrow iR = \varepsilon - \frac{q}{C} \Rightarrow i = \frac{\varepsilon}{R} - \frac{q}{RC} \Rightarrow \frac{dq}{dt} = \frac{\varepsilon}{R} - \frac{q}{RC}$$

$$\Rightarrow \frac{dq}{dt} = \frac{1}{RC}(\varepsilon C - q) \Rightarrow \frac{dq}{dt} = -\frac{1}{RC}(q - \varepsilon C)$$

(The above step is the one I find students forget most often. Yes, factor out a negative one on the right-hand side of the equation. Write it down. Remember it. No, it is not an obvious step you need to take.)

$$\Rightarrow \left(\frac{1}{q - \varepsilon C}\right) dq = -\frac{1}{RC} dt \Rightarrow \int_0^q \left(\frac{1}{q - \varepsilon C}\right) dq = -\int_0^t \left(\frac{1}{RC}\right) dt = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow [\ln(q - \varepsilon C)]_0^q = [\ln(q - \varepsilon C)] - [\ln(0 - \varepsilon C)] = \ln\left(\frac{q - \varepsilon C}{-\varepsilon C}\right) = -\frac{t}{RC}$$

$$\Rightarrow e^{\left[\ln\left(\frac{q - \varepsilon C}{-\varepsilon C}\right)\right]} = e^{-\frac{t}{RC}} \Rightarrow \frac{q - \varepsilon C}{-\varepsilon C} = e^{-\frac{t}{RC}} \Rightarrow q - \varepsilon C = (-\varepsilon C)\, e^{-\frac{t}{RC}}$$

$$\Rightarrow q = \varepsilon C - \varepsilon C e^{-\frac{t}{RC}} \Rightarrow q(t) = \varepsilon C\left(1 - e^{-\frac{t}{RC}}\right) \Rightarrow q(t) = q_{max}\left(1 - e^{-\frac{t}{RC}}\right)$$

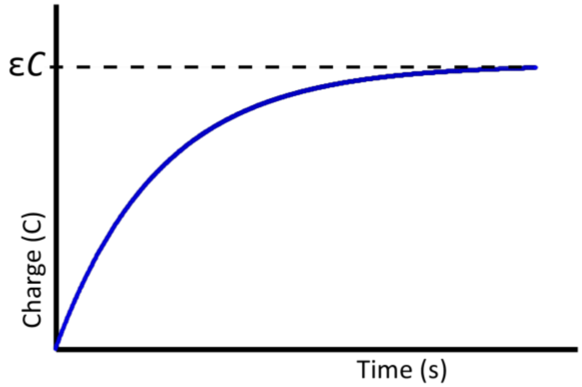
Applicable known equations:

$$\ln x - \ln y = \ln\left(\frac{x}{y}\right) \;\; \& \;\; e^{\ln x} = x$$

Notice this equation fits our limits for charge:

$$q(0) = \varepsilon C\left(1 - e^{-\frac{0}{RC}}\right) = \varepsilon C\left(1 - e^{0}\right) = \varepsilon C(1 - 1) = 0$$

$$q(\infty) = \varepsilon C\left(1 - e^{-\frac{\infty}{RC}}\right) = \varepsilon C\left(1 - e^{-\infty}\right) = \varepsilon C(1 - 0) = \varepsilon C = q_{max}$$



And we can derive the current through the circuit as a function of time:

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(\varepsilon C - \varepsilon C e^{-\frac{t}{RC}}\right) = \frac{d}{dt}\left(-\varepsilon C e^{-\frac{t}{RC}}\right) = -\varepsilon C \frac{d}{dt}\left(e^{-\frac{t}{RC}}\right) = -\varepsilon C\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}}$$

$$\Rightarrow i(t) = \left(\frac{\varepsilon}{R}\right)e^{-\frac{t}{RC}} \Rightarrow i(t) = i_{\text{max}}e^{-\frac{t}{RC}}$$

Again, this fits our limits for current:

$$\Rightarrow i(0) = \left(\frac{\varepsilon}{R}\right)e^{-\frac{0}{RC}} = \frac{\varepsilon}{R} = i_{\text{max}} \quad \& \quad \Rightarrow i(\infty) = \left(\frac{\varepsilon}{R}\right)e^{-\frac{\infty}{RC}} = 0$$