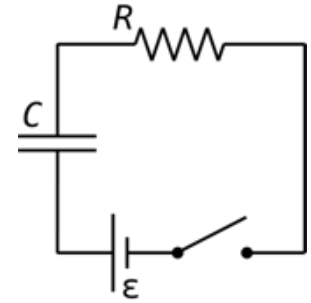
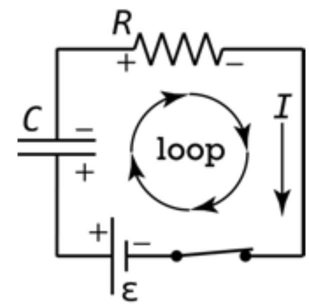
Flipping Physics Lecture Notes:
RC Circuits
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-rc-circuits.html



Up until this point we have assumed all changes in electric current, electric potential difference, and charge on capacitor plates were instantaneous. Today, we put a resistor and a capacitor together and learn how those variables change as a function of time. This is called an RC circuit. We start with a circuit composed of an uncharged capacitor, a resistor, a battery, and an open switch, all connected in series.

At time initial, $t_i = 0$, we close the switch.
We are *charging a capacitor through a resistor.*

Let's start by adding a loop in the direction of current flow in the circuit. Then use Kirchhoff's Loop Rule starting in the lower right-hand corner of the circuit:



$$\Delta V_{\text{loop}} = 0 = +\varepsilon - \Delta V_C - \Delta V_R$$

We can use the definition of capacitance to solve for the electric potential difference across the capacitor:

$$C = \frac{Q}{\Delta V} \Rightarrow \Delta V_C = \frac{Q}{C}$$

And we know Ohm's law: $\Delta V_R = IR$

$$\Rightarrow \Delta V_{\text{loop}} = 0 = \varepsilon - \frac{q}{C} - iR$$

Notice we are using lowercase "q" for charge because the charge is changing as a function of time. I wish we had a similar notation for current, I, however, if we used lowercase "i", I am sure it would be more confusing. So, please realize charge, q, and current, I, are both changing as a function of time in the above equation.

Now let's look at limits, starting with $t_i = 0$:

- The initial charge on the capacitor is zero: $q_i = 0$
- This means the initial electric potential difference across the capacitor is also zero:

$$\Rightarrow \Delta V_{C_i} = \frac{q}{C} = \frac{0}{C} = 0$$

- We can now use the loop equation to solve for the initial current through the circuit.

$$\Rightarrow 0 = \varepsilon - \frac{0}{C} - i_{\text{initial}}R \Rightarrow i_{\text{initial}}R = \varepsilon \Rightarrow i_{\text{initial}} = \frac{\varepsilon}{R} = i_{\text{max}}$$

- Because the charge on the capacitor will increase as a function of time, electric potential difference across the capacitor will also increase. This means the current in the circuit will decrease. In other words, the initial current in the circuit is also the maximum current.

And now the limit of "after a long time" or the $t_f \approx \infty$.

- The final current in the circuit is zero: $i_{\text{final}} \approx 0$

- This means the final electric potential difference across the resistor is also zero:
  $$\Rightarrow \Delta V_{R_f} = i_{\text{final}} R = (0)R = 0$$
- And we can use the loop equation to solve for the final charge on the capacitor:
  $$\Rightarrow 0 = \varepsilon - \frac{q_f}{C} - (0)R \Rightarrow q_f = \varepsilon C = q_{\max}$$
- Because we know the charge has been increasing this whole time, we know this is the maximum charge on the capacitor.

Now let's figure out what happens between $t_i = 0$ and $t_f \approx \infty$, however, before we do, I want to point out that AP Physics C: Electricity and Magnetism students are responsible for knowing how to derive these equations. So, yes, you do need to understand these derivations and be able to do them on your own. And here we go … starting with our Kirchhoff's Loop Rule equation:

$$0 = \varepsilon - \frac{q}{C} - iR \Rightarrow iR = \varepsilon - \frac{q}{C} \Rightarrow i = \frac{\varepsilon}{R} - \frac{q}{RC} \Rightarrow \frac{dq}{dt} = \frac{\varepsilon}{R} - \frac{q}{RC}$$

$$\Rightarrow \frac{dq}{dt} = \frac{1}{RC}(\varepsilon C - q) \Rightarrow \frac{dq}{dt} = -\frac{1}{RC}(q - \varepsilon C)$$

(The above step is the one I find students forget most often. Yes, factor out a negative one on the right-hand side of the equation. Write it down. Remember it. No, it is not an obvious step you need to take.)

$$\Rightarrow \left(\frac{1}{q - \varepsilon C}\right) dq = -\frac{1}{RC} dt \Rightarrow \int_0^q \left(\frac{1}{q - \varepsilon C}\right) dq = -\int_0^t \left(\frac{1}{RC}\right) dt = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow [\ln(q - \varepsilon C)]_0^q = [\ln(q - \varepsilon C)] - [\ln(0 - \varepsilon C)] = \ln\left(\frac{q - \varepsilon C}{-\varepsilon C}\right) = -\frac{t}{RC}$$

$$\Rightarrow e^{\left[\ln\left(\frac{q - \varepsilon C}{-\varepsilon C}\right)\right]} = e^{-\frac{t}{RC}} \Rightarrow \frac{q - \varepsilon C}{-\varepsilon C} = e^{-\frac{t}{RC}} \Rightarrow q - \varepsilon C = (-\varepsilon C)\, e^{-\frac{t}{RC}}$$

$$\Rightarrow q = \varepsilon C - \varepsilon C e^{-\frac{t}{RC}} \Rightarrow q(t) = \varepsilon C\left(1 - e^{-\frac{t}{RC}}\right) \Rightarrow q(t) = q_{\max}\left(1 - e^{-\frac{t}{RC}}\right)$$

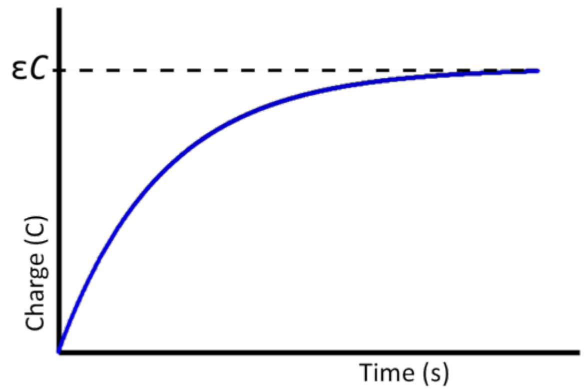
Applicable known equations:

$$\ln x - \ln y = \ln\left(\frac{x}{y}\right) \ \ \& \ \ e^{\ln x} = x$$

Notice this equation fits our limits for charge:

$$q(0) = \varepsilon C\left(1 - e^{-\frac{0}{RC}}\right) = \varepsilon C\left(1 - e^{0}\right) = \varepsilon C(1 - 1) = 0$$

$$q(\infty) = \varepsilon C\left(1 - e^{-\frac{\infty}{RC}}\right) = \varepsilon C\left(1 - e^{-\infty}\right) = \varepsilon C(1 - 0) = \varepsilon C = q_{\max}$$



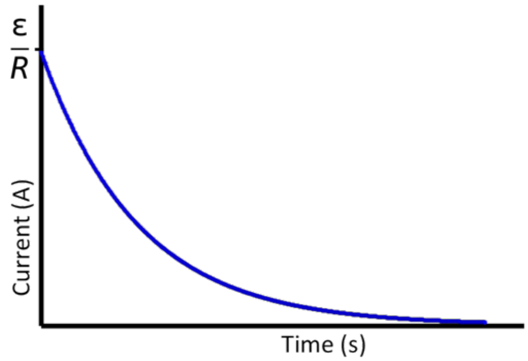
And we can derive the current through the circuit as a function of time:

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(\varepsilon C - \varepsilon C e^{-\frac{t}{RC}}\right) = \frac{d}{dt}\left(-\varepsilon C e^{-\frac{t}{RC}}\right) = -\varepsilon C \frac{d}{dt}\left(e^{-\frac{t}{RC}}\right) = -\varepsilon C\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}}$$

$$\Rightarrow i(t) = \left(\frac{\varepsilon}{R}\right)e^{-\frac{t}{RC}} \Rightarrow i(t) = i_{max}e^{-\frac{t}{RC}}$$

Again, this fits our limits for current:

$$\Rightarrow i(0) = \left(\frac{\varepsilon}{R}\right)e^{-\frac{0}{RC}} = \frac{\varepsilon}{R} = i_{max}$$ & $$\Rightarrow i(\infty) = \left(\frac{\varepsilon}{R}\right)e^{-\frac{\infty}{RC}} = 0$$



And now we get to talk about the time constant!

In the equations for charge and current as functions of time, there appears this expression: $e^{-\frac{t}{RC}}$

The time constant equals whatever appears in the denominator of that fraction. In other words, for an RC circuit, the time constant equals resistance times capacitance. The symbol for the time constant is the lowercase Greek letter tau, τ: $\tau = RC$

Before we discuss further what the times constant is, let's determine its units:

$$\tau = RC \Rightarrow \Omega F = \left(\frac{V}{A}\right)\left(\frac{C}{V}\right) = \frac{C}{A} = \frac{C}{\frac{C}{s}} = \frac{1}{\frac{1}{s}} = s$$

$$R = \frac{\Delta V}{I} \Rightarrow \Omega = \frac{V}{A} \ \& \ C = \frac{Q}{\Delta V} \Rightarrow F = \frac{C}{V} \ \& \ I = \frac{dq}{dt} \Rightarrow A = \frac{C}{s}$$

The units for the time constant are seconds; it is the *time* constant.

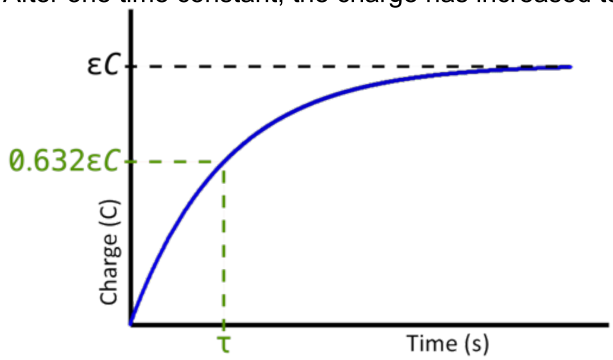
Let's replace RC with the time constant in our charge equation:

$$q(t) = q_{max}\left(1 - e^{-\frac{t}{RC}}\right) \Rightarrow q(t) = q_{max}\left(1 - e^{-\frac{t}{\tau}}\right)$$

And determine the charge on the capacitor after one time constant:

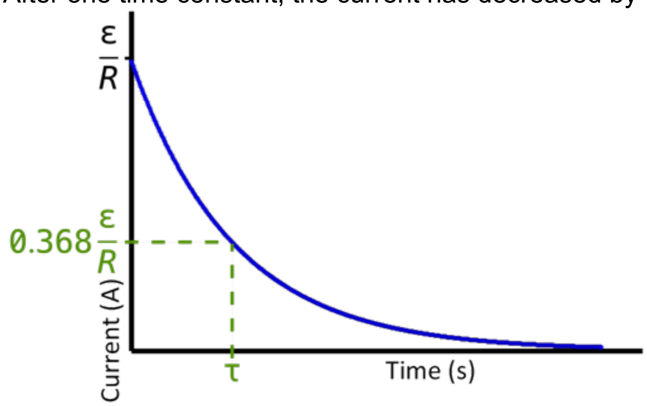
$$q(\tau) = q_{max}\left(1 - e^{-\frac{\tau}{\tau}}\right) = q_{max}\left(1 - e^{-1}\right) = q_{max}(1 - 0.368) \Rightarrow q(\tau) = 0.632 q_{max}$$

After one time constant, the charge has increased to 63.2% of its maximum value.



$$i(t) = i_{max} e^{-\frac{t}{\tau}} \Rightarrow i(\tau) = i_{max} e^{-\frac{\tau}{\tau}} = i_{max} e^{-1} \Rightarrow i(\tau) = 0.368 i_{max}$$

After one time constant, the current has decreased by 63.2% from its maximum value.



The time constant is the time it takes for a change of 63.2%. If you want to know more about the time constant, I talk about it in more detail in my video *Time Constant and the Drag Force*: https://www.flippingphysics.com/drag-force-time-constant.html

There are similar equations for discharging a capacitor through a resistor which we are not going to derive today.

Please realize the following two calculus equations are on the AP Equation Sheet:

$$\int \frac{dx}{x+a} = \ln|x+a| \ \& \ \frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$$