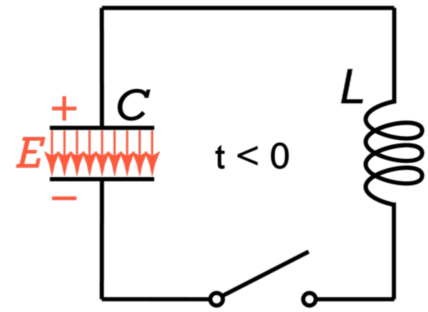
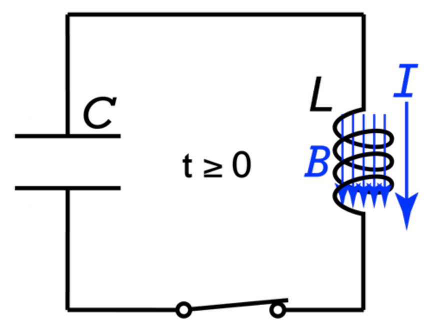
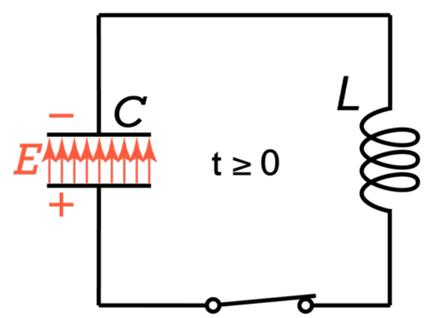
Flipping Physics Lecture Notes:
LC Circuits
Review for AP Physics C: Electricity and Magnetism
http://www.flippingphysics.com/apcem-lc-circuits.html







This LC circuit is a circuit with a capacitor, an inductor, and a switch. Before time t = 0, the switch was open for a long time. At time t = 0, the switch is closed and remains closed. A few general things to realize:

- The initial charge on the capacitor must be nonzero, if it were zero, nothing would happen when the switch is closed.
- The initial current in the circuit must be zero because there was no current in the open circuit before the switch was closed.
- The inductor opposes the change in current in the circuit which is why it takes time for the current to change from zero.
- The current through the inductor is from the charges leaving the capacitor to flow through the circuit, therefore, as current through the inductor increases, charge on the capacitor decreases.
- The electric field in the capacitor is decreasing in magnitude and the magnetic field in the inductor is increasing in magnitude.
- Once the charge is completely discharged, q = 0, the inductor has its maximum magnitude magnetic field and the current through the inductor is at its maximum.
- Current will continue to flow and build up charges on the plates of the capacitor, however, the orientation of the positive and negative plates will be reversed, and the current is decreasing.
- Eventually the current through the inductor will reduce to zero and charge will be at a maximum on the plates of the capacitor.
- Repeat the whole cycle in reverse.
- This is *simple harmonic motion!*
    - A horizontal mass-spring system is a good analogous situation.

Now let's derive equations for the LC Circuit, starting with the total energy in the circuit:

$$U_t = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2}LI^2 = \frac{q^2}{2C} + \frac{1}{2}Li^2$$

- Typically, we use uppercase symbols for constants and lowercase symbols for variables.
- We know $I_{\text{max}} \rightarrow q = 0 \;\&\; Q_{\text{max}} \rightarrow i = 0$
- We can take the derivative with respect to time of the total energy equation. We know the derivative of total energy in the LC circuit equals zero because these are all ideal components with zero resistance. In other words, no energy is being dissipated from the system.

$$\Rightarrow \frac{dU_t}{dt} = \frac{d}{dt}\left(\frac{q^2}{2C} + \frac{1}{2}Li^2\right) = 0$$

- We need to use the chain rule for both energy expressions because time is not a variable in either energy expression, however, both charge, q, and current, i, are changing with respect to time.

$$\Rightarrow 0 = \frac{d}{dt}\left(\frac{q^2}{2C}\right) + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) \Rightarrow 0 = \left(\frac{2q}{2C}\right)\frac{dq}{dt} + \left(\frac{2Li}{2}\right)\frac{di}{dt}$$

$$\Rightarrow 0 = \left(\frac{q}{C}\right)\frac{dq}{dt} + (Li)\frac{di}{dt} \ \ \& \ \ i = \frac{dq}{dt} \ \ \& \ \ \frac{di}{dt} = \frac{d^2q}{dt^2}$$

$$\Rightarrow 0 = \left(\frac{q}{C}\right)i + (Li)\frac{d^2q}{dt^2} = \frac{q}{C} + (L)\frac{d^2q}{dt^2} \Rightarrow -\frac{q}{C} = (L)\frac{d^2q}{dt^2} \Rightarrow \frac{d^2q}{dt^2} = -\frac{1}{LC}q$$

- The equation definition for simple harmonic motion is: $\frac{d^2x}{dt^2} = -\omega^2 x$
- Therefore, we know the angular frequency of an LC circuit And we can determine the period of an LC Circuit:

$$\Rightarrow \omega_{LC}^2 = \frac{1}{LC} \Rightarrow \omega_{LC} = \frac{1}{\sqrt{LC}}$$

$$\& \ \ \omega = 2\pi f = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} \Rightarrow T_{LC} = \frac{2\pi}{1/\sqrt{LC}} \Rightarrow T_{LC} = 2\pi\sqrt{LC}$$

- And we know a general equation which satisfies the simple harmonic motion equation:

$$x(t) = A\cos(\omega t + \phi) \Rightarrow q(t) = Q_{max}\cos\left(\frac{t}{\sqrt{LC}} + \phi\right) \Rightarrow q(t) = Q_{max}\cos\left(\frac{t}{\sqrt{LC}}\right)$$

- For this specific LC circuit the initial charge on the capacitor is Q_{max} , therefore, the phase constant is zero.

$$\& \ \ i = \frac{dq}{dt} \Rightarrow i(t) = \frac{d}{dt}\left[Q_{max}\cos\left(\frac{t}{\sqrt{LC}}\right)\right] = -Q_{max}\sin\left(\frac{t}{\sqrt{LC}}\right)\frac{d}{dt}\left(\frac{t}{\sqrt{LC}}\right)$$

- We can also determine current in an LC circuit as a function of time and an equation relating current maximum to charge maximum.

$$\Rightarrow i(t) = -\frac{Q_{max}}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right) \Rightarrow I_{max} = \frac{Q_{max}}{\sqrt{LC}} \Rightarrow i(t) = -I_{max}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

- We can also derive the current maximum using the equation for total energy in the LC circuit.

$$U_t = U_C + U_L = \frac{q^2}{2C} + \frac{1}{2}Li^2 = \frac{Q_{max}^2}{2C} + 0 = 0 + \frac{1}{2}LI_{max}^2 \Rightarrow \frac{Q_{max}^2}{C} = LI_{max}^2$$

$$\Rightarrow I_{max}^2 = \frac{Q_{max}^2}{LC} \Rightarrow I_{max} = \frac{Q_{max}}{\sqrt{LC}}$$

- We can determine equations for energy as functions of time.

$$U_C = \frac{q^2}{2C} \Rightarrow U_C(t) = \frac{\left[Q_{max}\cos\left(\frac{t}{\sqrt{LC}}\right)\right]^2}{2C} \Rightarrow U_C(t) = \frac{Q_{max}^2}{2C}\cos^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$U_L = \frac{1}{2}Li^2 \Rightarrow U_L(t) = \frac{1}{2}L\left[-\frac{Q_{max}}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)\right]^2 = \left(\frac{1}{2}L\right)\left(\frac{Q_{max}^2}{LC}\right)\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$\Rightarrow U_L(t) = \frac{Q_{max}^2}{2C}\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$U_t(t) = U_C(t) + U_L(t) = \frac{Q_{\max}^2}{2C}\cos^2\left(\frac{t}{\sqrt{LC}}\right) + \frac{Q_{\max}^2}{2C}\sin^2\left(\frac{t}{\sqrt{LC}}\right)$$

$$\Rightarrow U_t(t) = \left(\frac{Q_{\max}^2}{2C}\right)\left[\cos^2\left(\frac{t}{\sqrt{LC}}\right) + \sin^2\left(\frac{t}{\sqrt{LC}}\right)\right] \Rightarrow U_t(t) = \frac{Q_{\max}^2}{2C} \quad \& \quad \sin^2\theta + \cos^2\theta = 1$$

Below are two screenshots of the LC circuit animation. Honestly, you need to watch and hear the discussion of everything going on the animation to understand it.