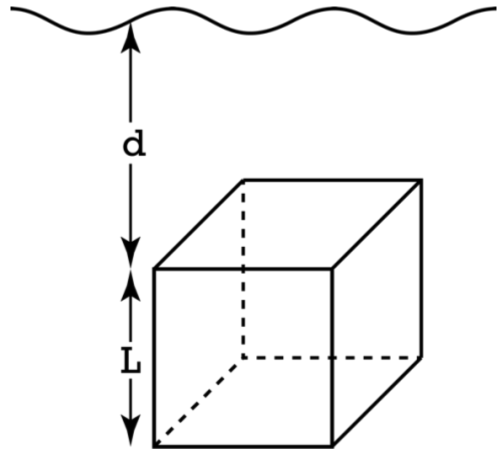
Flipping Physics Lecture Notes:
Buoyant Force Equation: Step-by-Step Derivation
http://www.flippingphysics.com/buoyant-force-derivation.html




Let's start by imagining a cube as shown with sides of width, L, at a depth, d, in a fluid of known density, $\rho_f$. Now let's determine the net force acting on this cube.

Knowns: Top of Cube Depth $= d$, Sides $= L$, $\rho_f$; $\sum F = ?$

In our lesson about fluid pressure, we determined that with increased depth in a fluid comes increased pressure caused by the weight of the fluid above. The net pressure from the fluid acting on the cube comes from the pressure acting inward on all six sides of the cube. Therefore, the force caused by this pressure on each side of the cube is:

- On the left side of the cube, the force acts to the right, which is positive.
- On the right side of the cube, the force acts to the left, which is negative.
- On the back of the cube, the force acts forward, which is positive.
- On the front of the cube, the force acts backward, which is negative.
- On the bottom of the cube, the force acts upward, which is positive.
- On the top of the cube, the force acts downward, which is negative.

$$\sum F = F_{\text{left}} - F_{\text{right}} + F_{\text{back}} - F_{\text{front}} + F_{\text{bottom}} - F_{\text{top}}$$

Notice the force acting on all four vertical sides of the cube are equal in magnitude because they are at equal depths in the fluid, therefore, the forces from the left and right sides cancel one another out and the forces from the back and front sides cancel one another out. That means the net force from the fluid acting on the cube comes from just the bottom and top sides of the cube.

$$F_{\text{left}} - F_{\text{right}} = 0 = F_{\text{back}} - F_{\text{front}} \Rightarrow \sum F = F_{\text{bottom}} - F_{\text{top}}$$

(Technically, because pressure increases with depth, the force also increases with depth, therefore the force is not constant on the vertical sides; it is larger near the bottom of each vertical side than near the top. However, due to symmetry, the forces at the same depth on each mirrored side have the same magnitude, and therefore cancel out.)

From Billy's first dream, we know the equation for pressure: $P = \frac{F_\perp}{A} \Rightarrow F_\perp = PA$

We can substitute that into the net force equation:

$$\Rightarrow \sum F = P_{\text{bottom}} A_{\text{bottom}} - P_{\text{top}} A_{\text{top}}$$

In the same lesson about fluid pressure, we derived the equation for the absolute pressure at a depth in a fluid. And we know the area of the bottom and top of the cube are the same. Let's identify that as the area of a side.

$$A_{\text{bottom}} = A_{\text{top}} = A_{\text{side}} \;\&\; P_{\text{absolute}} = P_0 + \rho g h$$

Where $P_0$ is the pressure pushing down on the top surface of the fluid and $\rho gh$ is the gauge pressure caused by the weight of the fluid above. We can substitute this equation in for the pressure at the bottom and top of the cube.

$$\Rightarrow \sum F = (P_0 + \rho_f g h_{\text{bottom}}) A_{\text{side}} - \left(P_0 + \rho_f g h_{\text{top}}\right) A_{\text{side}}$$

$$\Rightarrow \sum F = P_0 A_{\text{side}} + \rho_f g h_{\text{bottom}} A_{\text{side}} - P_0 A_{\text{side}} - \rho_f g h_{\text{top}} A_{\text{side}}$$

$$\Rightarrow \sum F = \rho_f g A_{\text{side}} \left(h_{\text{bottom}} - h_{\text{top}}\right) = \rho_f g A_{\text{side}} \left((d + L) - d\right)$$

$$\Rightarrow \sum F = \rho_f g A_{\text{side}} L = \rho_f g V_{\text{cube}} = \rho_f g V_f$$

Realize the volume of the cube equals the length of one side, L, times the area of one side. And remember the density in this pressure equation is the density of the fluid the cube is submerged in. We typically refer to that as the “fluid displaced” by the cube. And, from our lesson on density, we can recall that density equals mass divided by volume. So, we can substitute the mass of the fluid displaced by the cube in for the density of the fluid displaced by the cube times the volume of the fluid displaced by the cube.

$$\rho = \frac{m}{V} \Rightarrow m_f = \rho_f V_f \Rightarrow \sum F = m_f g = F_{\text{buoyant}}$$

And we get an equation for the net force acting on the cube caused by the fact that the cube is displacing the fluid. That net force acts upward on the cube and is called the buoyant force, $F_B$.

Buoyant Force:

- Upward force acting on an object in fluid.
- Equal in magnitude to the weight of the fluid displaced by the object.
    - The upward buoyant force acting on a hypothetical cube of water is equal in magnitude to the force of gravity acting on the hypothetical cube of water and causes our hypothetical cube of water to remain at rest.
- Sum of all the forces applied by the fluid surrounding the object.
- The composition of the object submerged in the fluid does not affect the buoyant force, only the mass of the fluid displaced by the object and the gravitational field.