Flipping Physics Lecture Notes:
Calculating Buoyant Force on a Submerged Sphere
http://www.flippingphysics.com/buoyant-force-steel.html

Previously we determined the buoyant force equals the weight of the fluid displaced by an object. Today let's look at a specific submerged steel sphere example.

Example: A steel sphere has a diameter of 50.7 mm and a density of 7,750 kg/m^3. What buoyant force acts on the steel sphere when it is submerged in water? The density of water is 1.00 x 10^3 kg/m^3.

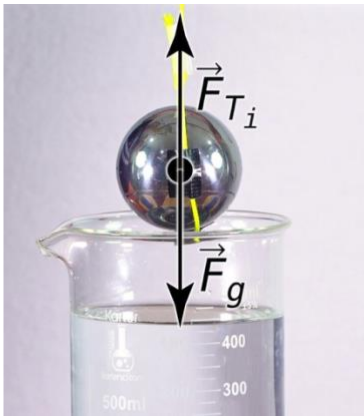
Knowns: $D = 50.7mm\left(\frac{1m}{1000mm}\right) = 0.0507m; R = \frac{D}{2} = \frac{0.0507}{2} = 0.02535m$

$\rho_{\text{steel}} = \rho_o = 7,750\frac{kg}{m^3}; \rho_{water} = \rho_f = 1.00 \times 10^3\frac{kg}{m^3}; F_B = ?$

$\& \ V_o = V_f = V_{\text{sphere}} = \frac{4}{3}\pi R^3 \ \& \ F_B = m_f g \ \& \ \rho = \frac{m}{V} \Rightarrow m = \rho V$

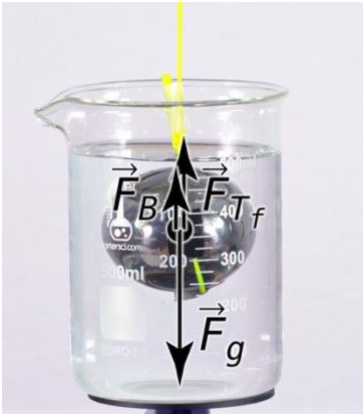
$\Rightarrow F_B = \rho_f V_f g = \rho_f\left(\frac{4}{3}\pi R^3\right)g = (1000)\left(\frac{4}{3}\pi(0.02535)^3\right)(9.81)$

$\Rightarrow F_B = 0.66941 \approx 0.669N$



For a moment, let's go back a step and just look at the steel sphere hanging on a string attached to a force sensor before we lower the steel sphere into the water. In the free body the force of gravity is down and the force of tension is up. We can solve for the initial tension in the string:

$$\sum F_{y_i} = F_{T_i} - F_g = m_o a_y = m_o(0) = 0$$
$$\Rightarrow F_{T_i} = F_g$$



When we lower the steel sphere into the water we add an upward buoyant force to the free body diagram and we can solve for the final tension in the string:

$$\sum F_{y_f} = F_{T_f} + F_B - F_g = m_o a_y = m_o(0) = 0$$
$$\Rightarrow F_{T_f} = F_g - F_B$$

In other words, the change in the force of tension measurement when I lower the steel sphere into the water is the negative of the buoyant force.

$$\Delta F_T = F_{T_f} - F_{T_i} = \left(F_g - F_B\right) - F_g = -F_B$$

That means, if I zero the force sensor before lowering the steel sphere into the water, the force sensor should read -0.67 N after I lower the steel sphere into the water.
And it equals -0.68 N. Why? Well, we did not include the string in any of our calculations. The string displaces a very small amount of water and has a very small buoyant force acting on it which increases the measured buoyant force by a small amount.

But what happens if I have the beaker and water resting on a force platform and I zero out the reading on the force platform before I lower the steel sphere into the water and then I lower the steel sphere into the water? What will the reading on the force platform be after I lower the steel sphere into the water?

0.68 N.

Because of Newton’s Third Law. Every force is an interaction between two objects. The upward buoyant force acts from the water on the sphere. That means there is an equal but opposite downward buoyant force which acts from the sphere on the water. That is why the force platform registers the same magnitude force as the force buoyancy.

And now, with the steel sphere in the water, let’s mark the water level in the beaker. And then take the steel sphere out of the water and add water until the water level is back to where it was with the steel sphere in the water. Again, the force platform registers 0.68 N because the buoyant force equals the weight of the fluid displaced by an object.

And we can even look at it in terms of pressure: $P = \frac{F_\perp}{A} \Rightarrow F = PA$

And the pressure at the bottom of a fluid is the gauge pressure:

$$P_{\text{gauge}} = \rho g h \Rightarrow F = (\rho g h) A$$

But we are really interested in the change in the force measured by the force sensor:

$$\Delta F = F_f - F_i = \rho g h_f A - \rho g h_i A = \rho g A (h_f - h_i) = \rho g A \Delta h$$

So, because the density, gravitational field, and cross-sectional area of the beaker are constants, and the change in depth of the water is the same for both the beaker with the steel sphere and the beaker without, the change in the force measurement is the same.

And because area of the base times height is volume:

$$\Rightarrow \Delta F = \rho g \Delta V = m_{\text{added water}} g = m_f g = F_B$$

We just proved the change in force measured by the force sensor equals the buoyant force acting on the steel sphere and it equals the weight of the volume of water added to the beaker after we removed the steel sphere.